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Continuous itinerary functions and dendrite maps

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Abstract

We examine the "kneading sequence" theory of maps on dendrites, concentrating on those maps having one "turning point" and the "unique itinerary property" (i.e., distinct points have distinct itineraries). This theory has major overlaps with the theories of polynomial Julia Sets and Hubbard Trees, but also has significant differences from those two theories (for which the unique itinerary property does not always hold). We show that the unique itinerary property is a powerful property, and allows a simple classification of such dendrite maps with respect to their kneading sequences (up to conjugacy if there is no nontrivial invariant subdendrite).

One of the major tools introduced here is the continuous itinerary function. If one takes the set of all sequences of the symbols used to define the itineraries with respect to a partition of a space, there is a natural topology which forces the itinerary function (from the original space into the space of sequences of symbols) to be continuous, although this often leads to a non-Hausdorff itinerary topology. Despite this apparent drawback, we show that this itinerary topology is a useful tool for analyzing the dynamics of continuous maps on metric spaces.

Characterizations in terms of kneading sequences are given for topological properties of these maps, including various transitivity properties, (in)decomposability of inverse limits, and the existence of certain "piecewise linearizations" of such maps which are a natural generalization of "tent" maps on the interval. The itinerary topology provides a natural topology for the parameter space of all kneading sequences, a natural subspace of which will be shown to have a one-point compactification that is a dendrite, with an interesting connection to the Mandelbrot Set.

1. Introduction

One of the major tools in the study of dynamical systems on topological spaces has been the use of itineraries and kneading sequences to track the orbits of points in the dynamical system (see, e.g., [12,8,13]). The basic idea (defined in more detail below) is simple: Given a map on a space, and a partition of the space into a certain number of pieces (usually, but not necessarily, finite), each piece is assigned a certain symbol, and a point in the space can be assigned an *itinerary*, i.e., the infinite sequence of symbols obtained by looking at which pieces of the partition (in which order) are visited by the orbit of the point in question. The dynamics of the shift map in the symbol space can then be used to analyze the dynamics of the original map.

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One paper which introduced many of the basic ideas for the unit interval is the well-known paper of Metropolis et al. [12], in which it was shown, for example, that if the turning point of a unimodal map on the interval was used to divide the interval into three pieces L (the points to the left of the turning point), C (the turning point), and R (the points to the right of the turning point), then the itinerary of the turning point alone was enough to reveal much about the dynamics of the corresponding map. They also outlined a simple criterion that characterized exactly when a certain sequence of R's, L's, and C's could be realized as the itinerary of the turning point (called the "kneading sequence") of some unimodal function on the interval. If one takes a continuous map f on a tree T instead of the interval, having exactly one point t at which f is not locally one-to-one, with $T \setminus \{t\}$ having two components, and t is neither an endpoint nor a branching point of T, then one could label t by the symbol C and the two components of $T \setminus \{t\}$ by L and R, leading to kneading sequences that are not possible for unimodal interval maps. Such kneading sequences have been most notably studied in the theory of Hubbard Trees, which have been used to analyze the structure of the Mandelbrot Set and the dynamics on Julia Sets of quadratic polynomials (although the symbols $1, \star, 0$ are typically used in place of L, C, R in that setting) (see, e.g., [4,5,7]).

A more limited class of unimodal interval maps that has attracted interest (being generally easier to analyze) is the class of "tent" maps, i.e., those unimodal interval maps f for which $|f'(x)|$ is defined and constant for all values of x other than the turning point. If one adds the requirement that f is not invariant on any proper nondegenerate subcontinuum (to avoid the triviality of extending the interval at an endpoint to get a nonconjugate map), then it is easy to prove that two such tent maps are conjugate if and only if they have the same kneading sequence. This is a much stronger result than what is available on unimodal interval maps or on Hubbard Trees. On the other hand, it is well known that some kneading sequences which can occur for unimodal maps in general (e.g., those corresponding to period doublings) do not occur for any tent map. Note that a continuous function f on the unit interval I is a tent map if and only if there is a constant λ and a point t (the turning point) so that for any arc $A \subseteq I$ which misses t, the length of $f(A)$ is λ times the length of A. Worded in this way, we get a natural generalization of the definition to trees and dendrites (for which we shall use the term "tentlike" in the more formal definition below), provided that the topology is generated by a taxicab metric, so that there is a reasonable definition for the "length" of an arc. It is easy to see that such maps have another useful property, i.e., that different points have different itineraries (with more than three symbols sometimes needed to define the itineraries). This gives a natural weakening of the term "tentlike" (called "tentish", i.e., unimodal plus different points have different itineraries) which turns out to be the natural assumption which allows the most important parts of the theory to go through, including, for example, the result that if we add the hypothesis that there is no proper nondegenerate subcontinuum on which the function is invariant, then the kneading sequence alone will be enough to specify the map up to topological conjugacy. The properties of being "tentlike" or "tentish" will often be realized by certain quadratic complex polynomials restricted to their Julia sets, providing an interesting (but incomplete) overlap with the theory of Julia Sets and Hubbard Trees. (Of course, the Julia Set of such a map is often not even a dendrite.)

One of the natural annoyances that has been present in the theory of itineraries and kneading sequences is that if one uses the natural compact metric topology on the set of all sequences of R's, C's, and L's (or whatever other symbols are being used), i.e., the infinite product of the discrete topology on $\{R, C, L\}$, then the itinerary function will not be a continuous function unless the sets representing the symbols R, C, L are clopen sets, leading to discontinuous itinerary functions if the spaces are connected. Some other alternatives (such as using overlapping sets or having the symbol corresponding to C count as a "wild card") lead to ambiguous itineraries, so that there is no itinerary *function*. We show that there is a natural alternative approach, in which the set of all sequences of symbols is endowed with a different topology specifically designed to make the itinerary function continuous. This topology (called the "itinerary topology") will be obviously non-Hausdorff (but containing a homeomorphic copy of the Hilbert Cube) in the cases of interest here, but we shall have more than enough compensation for that apparent drawback, including the fact that with the itinerary topology, the itinerary function will be a *homeomorphism onto its range* in a surprisingly large number of cases. Even though the spaces themselves are non-Hausdorff, subspaces of these itinerary spaces defined in a combinatorially natural way will turn out to be dendrites, leading to a natural classification of dendrite maps having the property of being "tentish". This topology will also give a natural way of defining a topology on the parameter space, i.e., the set of all kneading sequences. While the topology of all kneading sequences will turn out to be non-Hausdorff in the itinerary topology, the set of all kneading sequences realized by "tentlike" maps will be a locally compact metric space whose one-point compactification is a dendrite, with an interesting connection to the Mandelbrot Set, under the assumption that the latter is locally connected (which is still an open problem).

The remainder of Section 1 introduces the main definitions, and proves a few of the more basic preliminary results. In Section 2, we examine the combinatorics of itineraries, showing that the subspace itinerary topology on certain natural sets of sequences is always a dendrite, leading to a classification of all "tentish" maps (i.e., all unimodal dendrite maps having the unique itinerary property). In Section 3, the combinatorics of itineraries is further examined, with discussions of the connections to Hubbard Trees, period n-tupling and renormalization. Section 4 contains a number of miscellaneous results showing how information about the map can be determined from combinatorial information in the kneading sequence. Section 5 examines the more complicated itinerary topology of the parameter space, showing that even though it is not Hausdorff, there is a natural Hausdorff subspace leading to a dendrite structure. Section 6 examines the problem of which "tentish" maps are "tentlike" maps, a problem that turns out to have close connections to the material in Section 5.

Definition 1.1. A *continuum* is a compact connected metric space. An *arc* is a space homeomorphic to the unit interval $[0, 1]$. A space X is *uniquely arcwise connected* iff for every distinct $x, y \in X$ there is exactly one arc $A \subseteq X$ having x and y as endpoints. A *tree* is a uniquely arcwise connected union of finitely many arcs. A continuum C is *treelike* iff for every $\varepsilon > 0$ there is a continuous function $f : C \to T$ for some tree T (which may depend on ε) such that $f^{-1}(x)$ has diameter less than ε for every $x \in T$. A *dendrite* is a uniquely arcwise connected locally connected continuum. A *dendroid* is a uniquely arcwise connected treelike continuum. It is well known that every tree is a dendrite and that every dendrite is a dendroid. A point x of a uniquely arcwise connected space X is called an *endpoint* of X if $X \setminus \{x\}$ is also arcwise connected, and we say that x is a *branching point* of X if $X \setminus \{x\}$ has at least three arc-components.

Definition 1.2. If C is a dendrite, and $A \subseteq C$, let $[A]$ be the smallest subcontinuum of C containing A, i.e., the closure of the union of all arcs having members of A as endpoints. If $x, y \in C$, we write $[x, y]$ $(= [y, x])$ for $[\{x, y\}]$ (an arc with x and y as endpoints if $x \neq y$), with $(x, y] = [y, x) = [x, y] \setminus \{x\}$, and $(x, y) = (y, x) = [x, y] \setminus \{x, y\}$, noting that (x, y) is not always open in the topology of C. A metric d on a dendrite D is called a *taxicab metric* if d is compatible with the topology of D and $d(x, z) = d(x, y) + d(y, z)$ whenever $y \in [x, z]$. It is routine to show that every dendrite admits a compatible taxicab metric, and that any uniquely arcwise connected continuum having such a compatible metric is a dendrite. Note that if d is a taxicab metric for D, then compactness implies that $\{d(x, y)\colon\ x, y \in D\}$ is bounded, and that there are points $a, b \in D$ such that $d(a, b)$ is the least upper bound of $\{d(x, y)\colon\ x, y \in D\}$. we define $M_d = \sup\{d(x, y)\colon\ x, y \in D\}$ for a taxicab metric d. If x, y, z are (not necessarily distinct) points in a uniquely arcwise connected space C, there will be exactly one point $w \in C$ such that $[w, x] \cap [w, y] = [w, x] \cap [w, z] = [w, y] \cap [w, z] = \{w\}$, and that point w will be called $w(x, y, z)$. Note that if $[a, b, c]$ is not an arc, then $w(a, b, c)$ is just the unique branching point of $[a, b, c]$, and that if C is a dendrite, then w will be a continuous function from C^3 into C (but w is not continuous for uniquely arcwise connected spaces in general). If C is a continuum and $X \subseteq C$, we will say that X is *arc-dense* in C if X intersects every arc in C. Note that if C is an arcwise connected continuum, then arc-dense implies dense, and that if C is a tree, then arc-dense is equivalent to dense.

Definition 1.3. As usual, $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ are the sets of positive integers, integers, real numbers, and complex numbers, respectively, and we let $\omega = \mathbb{N} \cup \{0\}$ denote the set of nonnegative integers. If $S \subseteq \mathbb{Z}$, then we let $Add(S)$ be the additive subgroup of $\mathbb{Z}$ generated by S, recalling the basic fact from Number Theory that if S is a nonempty subset of $\mathbb{N}$, then the greatest common divisor (abbreviated g.c.d.) of S is the smallest positive element of $Add(S)$. If $f : X \to X$ is a function, f^n is f composed with itself n times (with $f^0(x) = x$), and define $Orb_f(x) = \{f^n(x)\colon\ n \in \omega\}$ for $x \in X$. If $f|Orb_f(x)$ is a bijection, then x is said to be a *periodic point* of period n, where n is the number of points in $Orb_f(x)$ (which must be finite in this case). If $Orb_f(x)$ is finite, we say that x is *preperiodic* with respect to f, noting that we consider periodic points to be also preperiodic. Define $Pre_f(x) = \{y\colon\ f^n(y) = x \text{ for some } n \in \omega\}$ (i.e., the set of all eventual preimages of x).

Definition 1.4. We adopt the following notation concerning sequences. An *infinite sequence* (or just *sequence*) is a function whose domain is ω. A *finite sequence* of length n is a function whose domain is $\{0, 1, 2, \ldots, n - 1\}$ for some $n \in \omega$. A *bi-infinite sequence* is a function whose domain is the set of integers. If α is an infinite sequence, then the *shift* of α, denoted $\sigma(\alpha)$, is the infinite sequence β such that $\beta_n = \alpha_{n+1}$ for all $n \in \omega$. The terms *periodic* and *preperiodic* defined above, when applied to infinite sequences, are applied with respect to the shift function. If α and β are finite sequences of length m and n, respectively, then the *concatenation* of α and β, denoted by $\alpha\beta$, is the finite sequence γ

of length $m+n$ given in the obvious way: $\gamma_i = \alpha_i$ for $0 \leqslant i \leqslant m-1$, and $\gamma_i = \beta_{i-m}$ for $m \leqslant i \leqslant m+n-1$. Similarly, an infinite sequence may take part in a concatenation if it is the rightmost term. If $\alpha = \langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \rangle$ is a finite sequence, then $\overline{\alpha} = \overline{\langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \rangle}$ is the obvious infinite extension α' of α such that $\alpha'_{k+n} = \alpha'_k$ for all $k \geqslant 0$. If the sequence is integer-valued and the integers involved are all less than 10 (as they generally will be in the examples given), we omit the commas and angle brackets, and write, for example, 012 for $\langle 0, 1, 2 \rangle$, and $\overline{0112}$ for $\overline{\langle 0, 1, 1, 2 \rangle}$. Preperiodic sequences can easily be represented by combining this notation with concatenation, e.g., $012\overline{34} = \langle 0, 1, 2, 3, 4, 3, 4, 3, 4, 3, 4, \ldots \rangle$. Exponents will be used to indicate symbols (or groups of symbols) that are to be repeated a finite number of times, for example, $\overline{01^32}$ for $\overline{01112}$ and $\overline{01(12)^33}$ for $\overline{011212123}$. If α is a sequence (finite or infinite) of length at least n then $\alpha|n$ is defined to be the restriction of α to $\{0, 1, 2, \ldots, n-1\}$.

Definition 1.5. Let D be a dendrite, and let $f : D \to D$ be continuous. We say that f is *locally one-to-one* at a point $x \in D$ if there is a neighborhood U of x such that $f|U$ is one-to-one. A point at which f is not locally one-to-one will be called a *turning point* of f. [Note that if f is a continuous map on a dendrite, and $f(x) = f(y)$, then there must be a turning point in (x, y), so that f will be one-to-one on any connected subset containing no turning points.] A map from a dendrite into itself will be called *unimodal* iff it has at most one turning point. If $t \in D$, then we define the *legs* of D (with respect to t) to be the components of $D \setminus \{t\}$. Since a dendrite has only countably many legs, we may assume for convenience that each leg is labeled as L_n for some positive integer n, and we shall let $L_0 = \{t\}$, but L_0 will not be considered a leg. If the legs of D are enumerated as $L_1, L_2, \ldots$ in the order that they are visited by the orbit of t with respect to f, then we shall say that this enumeration satisfies the *labeling convention*. So, in this case, L_1 is the leg containing $f(t)$ (except in the uninteresting case when t is a fixed point), L_2 is the leg containing the first member $f^n(t)$ of the orbit of t not in $L_1 \cup \{t\}$ (if such n exists), and so forth. Legs not intersecting the orbit of t are then enumerated arbitrarily as L_n with labels n not already used by the legs which intersect the orbit of t. It will be easy to see that the exact labeling used will be unimportant in most cases. The labeling convention (or something like it) is needed occasionally (such as when we want a topological invariant), and is useful for limiting examples to only essentially different cases, but it will often be more convenient to do without it.

Definition 1.6. The *itinerary* ι of a point $x \in D$ with respect to the function f and turning point t is the sequence $\iota(x, f) = \langle \iota_0, \iota_1, \iota_2, \ldots \rangle$ of labels defined by $\iota_n = k$ iff $f^n(x) \in L_k$. If f has exactly one turning point, then the *kneading sequence* of the function f with respect to the labeling $L = \langle L_1, L_2, \ldots \rangle$, written $\tau^L(f)$, is defined as the sequence $\iota(t, f)$, where t is the unique turning point of f. If the legs are labeled according to the labeling convention, then we omit the subscript and write $\tau^L(f) = \tau(f)$. If f and t are obvious from context, we write $\iota(x)$ for $\iota(x, f)$ and τ for $\tau(f)$. If $\iota(x, f) \neq \iota(y, f)$ whenever $x \neq y$, then f is said to have the *unique itinerary property*.

Definition 1.7. An infinite sequence τ is said to satisfy the *labeling convention* if and only if its range is a subset of ω and whenever $\tau_i = n$ for some $n > 0$ there is a $j < i$ such that $\tau_j = n-1$. Note that if τ is any infinite sequence, then there is a unique one-to-one function $h : \mathit{range}(\tau) \to \omega$ such that $h \circ \tau$ satisfies the labeling convention, and that a kneading sequence will satisfy the labeling convention if and only if the legs were enumerated according to the labeling convention.

Note. Kneading sequences have often been defined as the itinerary of the image of the turning (or critical) point, rather than the itinerary of the turning point. Both contain the same amount of information, since the extra symbol appearing at the beginning in the latter case has only redundant data. We shall use the latter approach here, which turns out to provide more convenient indexing in many of the definitions and proofs, and is essential in some generalizations to more turning points where the extra symbol is not redundant. Converting to the other approach is a simple matter of dropping the redundant initial "0" (or whatever other symbol is used for the turning point).

Definition 1.8. Suppose $1 < \lambda \leqslant 2$, let a, b be such that $\frac{-1}{\lambda-1} \leqslant a \leqslant 1 - \lambda b$, $1 \leqslant b \leqslant \frac{1}{\lambda-1}$, and define $f : [a, b] \to [a, b]$ by $f(x) = 1 - \lambda|x|$, $a \leqslant x \leqslant b$. Such a function f will be called a *slope λ tent map* with turning point 0.

It is easy to check that the possible choices given for a and b are exactly those needed to guarantee that $0 \in [a, b]$ and that f maps $[a, b]$ into itself. Measuring itineraries with respect to the turning point $L_0 = \{0\}$, the legs are $L_1 = (0, b]$ and $L_2 = [a, 0)$ (corresponding respectively to the C, R and L of [12]), and it is easily seen that every tent map has the unique itinerary property, because the distance between points increases by a factor of λ with each application of f until the eventual images under f are in different legs (or one of them hits the turning point).

Definition 1.9. Let D be a dendrite, with $f : D \to D$ continuous. The map f is said to be *tentlike* with *expansion factor* $\lambda > 1$ if and only if f is unimodal with turning point t, and there exists a taxicab metric d inducing the topology of D such that if x and y are both in the same component of $D \setminus \{t\}$, $d(f(x), f(y)) = \lambda d(x, y)$.

Proposition 1.10. *If $f : D \to D$ is a tentlike dendrite map with turning point t and expansion factor λ, and f has the unique itinerary property with respect to t.*

Proof. If x and y have the same itinerary, then $d(f^n(x), f^n(y)) = \lambda^n d(x, y)$ for all positive integers x and y, and since a taxicab metric on a dendrite is bounded, we must have $d(x, y) = 0$. □

This observations leads us to a natural weakening of the definition of tentlike:

Definition 1.11. Let D be a dendrite, with $f : D \to D$ continuous. The map f is said to be *tentish* if and only if it is unimodal with turning point t and has the unique itinerary property with respect to t. Note that every tentlike dendrite map is necessarily tentish. We say that a function $f : X \to X$ is *expanding* on a subset S of a metric space (X, d) if there is a constant $\lambda > 1$ such that $d(f(x), f(y)) > \lambda d(x, y)$ whenever $x, y \in S$.

Proposition 1.12. *Every tentlike map of a dendrite is expanding on every leg. Furthermore, if $f : D \to D$ is a dendrite map, and t is a point of D such that f is expanding on every component of $D \setminus \{t\}$ (with respect to some metric compatible with the topology), then f is tentish.*

The proof of the second sentence is exactly the same as Proposition 1.10, and gives us a convenient way of constructing tentish maps on trees, where the hypothesis of Proposition 1.12 is easy to arrange.

Example 1.13. Let T be the tree with three endpoints t_1, t_2, t_3, branching point c. Put a taxicab metric on T so that $1 < d(c, t_1) < d(c, t_2) < d(c, t_3) = 2$, and let t_0 be the midpoint of $[c, t_2]$. Then it is easy to arrange a map f on T such that c and t_3 are fixed points, $f^n(t_n) = t_{n+1}$, $n = 0, 1, 2$, and f is expanding on each component of $T \setminus \{t_0\}$. This will be a tentish map with kneading sequence $\tau_f = \overline{0112}$. See Fig. 2, where the points are labeled by their itineraries, with τ^n as the nth shift of the kneading sequence τ.

The existence of tentish maps which are not tentlike is not immediately obvious. This will be shown in Section 6, where tentlike maps are discussed in detail, and a characterization is given (in terms of kneading sequences) for which tentish maps are tentlike. However, the unique itinerary property is all that is needed to get many of the most interesting results, and we shall concentrate on that property for the next couple of sections. We first give an example of a tentish map whose domain is not an interval, in fact, not even a tree.

Example 1.14. Modify Example 1.13 as follows. Let c_0 be the point in $[t_0, t_3]$ such that $f(c_0) = c$, an then use induction to define $c_n \in [t_0, t_3]$ with $f(c_n) = c_{n-1}$, noting that the c_n's converge to t_3. Add new arcs $[c_n, x_n]$ for $n \in \omega$, making sure that there lengths decrease to 0 so that the result will still be a dendrite. Extend the function f by mapping $[c_0, x_0]$ to $[c, t_2]$ and $[c_n, t_n]$ to $[c_{n-1}, x_{n-1}]$ for $n > 0$. This mapping will be a tentish dendrite map (with the same kneading sequence as before).

Variations of this trick are always available for any tentish map (but not always without increasing the number of legs). Examples are also abundant among Julia sets of polynomials.

Example 1.15. Let f be the restriction of the complex map $z^2 + i$ to the Julia set of $z^2 + i$. Then f is tentish, with $\tau(f) = 01\overline{12}$. It is known that this particular Julia set is a dendrite, and the fact that f is unimodal is a trivial consequence of the fact that the map $z^2 + i$ is locally one-to-one at all points other than 0. The fact that the unique itinerary property hold is more complicated, but follows from the standard theory of polynomial Julia sets, which gives

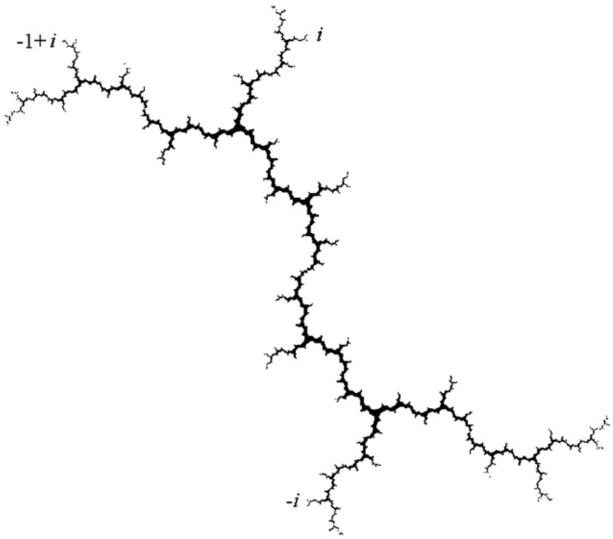

Fig. 1. The Julia Set of $z^2 + i$.

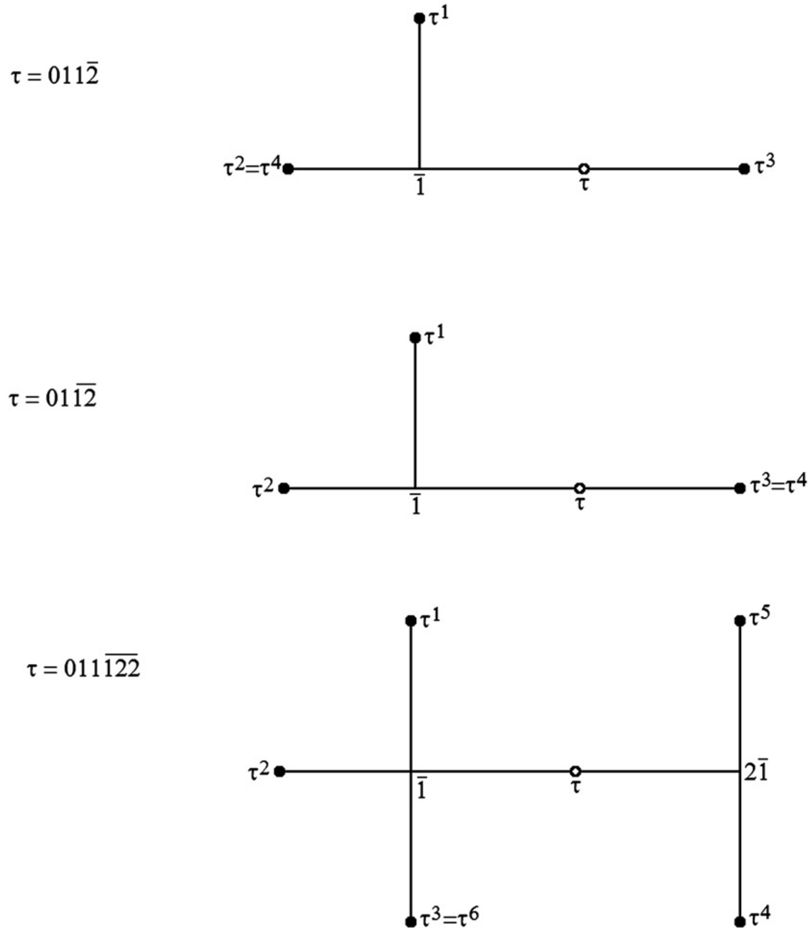

Fig. 2. Examples of tentish maps for various kneading sequences τ.

a semiconjugacy between f and the z^2 map on the unit circle in those cases where the Julia set is locally connected [see, e.g., [4]]. See Fig. 1, and note that the orbit of the turning point 0 is $0, i, -1+i, -i, -1+i, -i, \ldots$. Thus $[i, -1+i, -i]$ is an invariant tree, with kneading sequence $01\overline{12}$. See also Fig. 2.

There are also many additional such examples. Of course, the Julia set of z^2+c is often not a dendrite, but if the Julia set is a dendrite, then the restriction of z^2+c to the Julia set will be tentish (and often, but not always, tentlike).

Proposition 1.16. *If $f : D \to D$ is tentish, with turning point t, then for every leg L, f is one-to-one on $L \cup \{t\}$, and $f(t) \neq t$.*

Proposition 1.17. *If $f : D \to D$ is a tentish map with turning point t, then $Pre_f(t)$ is arc-dense in D.*

Proof. Given an arc $[x, y] \subseteq D$, let $n \in \omega$ be least such that either $f^n(x)$ and $f^n(y)$ are in different components of $D \setminus \{t\}$, or one of them is equal to t. Then $t \in f^n[x, y]$, so $f^{-n}(t)$ intersects $[x, y]$. □

Proposition 1.18. *If $f : D \to D$ and $g : E \to E$ are conjugate tentish maps (i.e., there is a homeomorphism $h : D \to E$ such that $h \circ f = g \circ h$), then $\tau(f) = \tau(g)$.*

Recall that the notation $\tau(f)$ (without the superscript $\tau^L(f)$) means that we are using the labeling convention. Note that the labeling convention (or something like it) is required for this result.

The following simple lemma plays a key role in the development of the itinerary topology in Section 2.

Lemma 1.19. *If $f : D \to D$ is tentish, and $x, y \in D$ are distinct, then there is an integer n such that $0 \neq \iota_n(x) \neq \iota_n(y) \neq 0$.*

Proof. By contradiction. Let x and y be distinct points of D, and suppose that there is no n as above. Then for each $n \in \omega$, $[f^n(x), f^n(y)]$ is contained in $L \cup \{t\}$ for some leg L, and f is therefore one-to-one on $[f^n(x), f^n(y)]$. Thus, every point in (x, y) has the same itinerary, contradicting the unique itinerary property. □

Corollary 1.20. *If $f : D \to D$ is tentish, then $\tau(f)$ begins with a finite sequence of the form 01^n2.*

Proof. By the previous lemma, the itinerary must contain a 2, for otherwise the points t and $f(t)$ would violate the conclusion of Lemma 2.39. Also, if $\tau(n) = 0$ for some $n > 0$, then i must be periodic (of period the least positive such n), so a 2 must occur in the itinerary before the second 0 (if there is one). By the convention that the legs were numbered in the order that they were visited by the orbit of the critical point, the result follows. □

Definition 1.21. If $f : D \to D$ is a tentish map, let $k(f)$ be the least integer k such that $\tau_k(f) = 2$.

Theorem 1.22. *Let $f : D \to D$ be tentish, with turning point t. Then there is a fixed point c of f in the interval $(t, f(t))$ such that $D \setminus \{c\}$ has exactly $k(f)$ components, each of which contains exactly one of the points $t_0, t_1, \ldots, t_{k(f)-1}$, where $t_n = f^n(t)$. No leg of D can contain more than one fixed point of f, and if there are any fixed points of f other than c, they must be endpoints of D.*

Proof. Let $k = k(f)$, and assume $k \geqslant 3$, the case $k = 2$ being an easy exercise. Let T be the tree $[t_0, t_1, \ldots, t_{k-1}]$, and let T' be the tree $[t_1, \ldots, t_k]$. Since $t_0 \in [t_1, t_k]$, we also have $T' = [t_0, \ldots, t_k]$ and $T \subseteq T'$. Note that since $T \subseteq L_1 \cup \{t_0\}$, f maps T homeomorphically onto T'. Since t_k is an endpoint of T', t_{k-1} is an endpoint of T, and therefore of T', t_0 being the only endpoint of T which is not an endpoint of T'. Using the same argument by backwards induction, the endpoints of T are exactly $t_0, \ldots, t_{k-1}$, and the endpoints of T' are exactly $t_1, \ldots, t_k$. Now, let $1 \leqslant i < j < k$. Then $w(t_i, t_j, t_k)$ (the branching point of $[t_i, t_j, t_k]$) must be in L_1 (since otherwise t_i and t_j would not be in the same component of $D \setminus \{t_0\}$), and thus $w(t_i, t_j, t_k) = w(t_i, t_j, t_0)$. This, along with the observation that $f(w(t_i, t_j, t_n) = w(t_{i+1}, t_{j+1}, t_{n+1})$ for $0 \leqslant i < j < n < k$, shows that the set $W = \{w(t_i, t_j, t_n)\colon 0 \leqslant i < j < n < k\}$ is closed under the function f. Since $W \subseteq L_1$, every element of W has itinerary $\overline{1}$, and thus the unique itinerary property implies that W is a singleton consisting of a single fixed point c. Thus, $T \setminus \{c\}$ has exactly k components, each of which contains exactly one of the points $t_0, t_1, \ldots, t_{k-1}$.

We now need to see that $D \setminus \{c\}$ has no additional components. Thus, let K be the union of all components of $D \setminus \{c\}$ which contain none of the points $t_0, t_1, \ldots, t_{k-1}$, and clearly $K \subseteq L_1$. Thus, no point of K could map to c, because that point would then have the same itinerary as c, violating the unique itinerary property. Thus, continuity demands that $f(K) \subseteq K$, and thus every point of K has itinerary $\overline{1}$. Since the unique itinerary property implies that no such points exist other than c, K must be empty. If d is any fixed point of f other than c, then clearly d has itinerary $\overline{m}$ for some $m \neq 1$. Furthermore, $[d, t_0] \subseteq [d, t_1]$. Thus, if we repeat the same argument on d that we just used for c, we get that $D \setminus \{d\}$ has a single component, i.e., d is an endpoint. □

Definition 1.23. If f is a tentish map, then the unique fixed point c of f in L_1 as given in the above result will be called the *central* fixed point of f, and will be denoted $c(f)$.

Lemma 1.24. *Let $f : D \to D$ be tentish, and for each $n \in N$ let $T_n = [\{t_j\colon 0 \leqslant j \leqslant n\}]$, where $t_n = f^n(t)$ (i.e., T_n is the tree generated by the part of the orbit of t up to $f^n(t)$). Suppose that for some $n > 0$, T_n has fewer than n endpoints. Then $T_m = T_{n-1}$ for all $m \geqslant n-1$.*

Proof. By the previous theorem, $T_{k(f)}$ has $k(f)$ endpoints, and it is also clear that T_n has $n+1$ endpoints for $1 \leqslant n \leqslant k(f)-1$. Suppose that T_n has fewer than n endpoints, and let n be least such that this happens. Since the number of endpoints of T_{m+1} is obviously either the same as or one more than the number of endpoints of T_m for any m, T_{n-1} has $n-1$ endpoints, as does T_n. Clearly, $n > k(f)$, and the $n-1$ endpoints of T_{n-1} are $\{t_j\colon 1 \leqslant j \leqslant n-1\}$. Aiming for a contradiction, suppose that $T_n \neq T_{n-1}$. This can happen only if there is an m $(0 < m < n)$ such that $t_m \in (t_0, t_n)$. Fix this m, and note that one-to-oneness of f on (t_0, t_n) implies that $t_{m+1} = f(t_m) \in (t_1, t_{n+1})$, so that $T_{n+1} = T_n \cup (t_{m+1}, t_{n+1}]$. Thus, $t_{m+1} \in (t_0, t_{n+1})$, and the same argument can then be repeated by induction to get that $t_{m+j} \in (t_0, t_{n+j})$ for all $j \geqslant 0$. But then t_m and t_n have the same itinerary, a contradiction. Thus $T_n = T_{n-1}$. However, this, plus the fact that f is one-to-one on each leg, implies that $f(T_n) = T_n$), so that the entire orbit of t is contained in T_n. □

Corollary 1.25. *Let $f : D \to D$ be tentish with turning point t, and suppose $n \in \omega$ is such that $\tau_n(f) \neq \tau_i(f)$ for all $i < n$. Then $f(t), f^2(t), \ldots, f^n(t)$ are all endpoints of $[Orb_f(t)]$. In particular, $f(t)$, $f^2(t), \ldots, f^k(t)$ are all endpoints of $[Orb_f(t)]$, where $k = k(f)$.*

Definition 1.26. A tentish map $f : D \to D$ with turning point t is said to be *minimally tentish* iff there is no proper subdendrite $E \subseteq D$ containing t which is closed under f (or, equivalently, $D = [Orb_f(t)]$). It is trivial to see that every tentish map f has a unique restriction that is minimally tentish, i.e., $f|[Orb_f(t)]$.

By a *pseudoleg* of D we mean any union of legs of D on which f is one-to-one. Note that the previous theorem implies that f is not one-to-one on the union of any two legs which intersect the orbit of t, so a pseudoleg can never contain more than one leg which intersects the orbit of t, because if $1 < i < j$ and $t \in (t_i, t_j)$, then f is not one-to-one on (t_i, t_j), since $t_1 \notin (t_{i+1}, t_{j+1})$. Given a partition of $D \setminus \{t\}$ into pseudolegs, we can enumerate the pseudolegs according to the same convention as was done for legs above, and define itineraries for points as before. Note that while the itinerary of some points may change, and may depend on which partition into pseudolegs is used, the itinerary of t does not change, so we have the same kneading sequence as before, *provided that we use the labeling convention to enumerate the legs and pseudolegs in both cases* (or, weaker, provided that each pseudoleg intersecting the orbit of t receives the same label as the contained leg which intersects that orbit). As the following result shows, this also does not affect the unique itinerary property.

Theorem 1.27. *If $f : D \to D$ is a continuous dendrite map, $t \in D$, $\langle L_n \rangle$ is an enumeration of the legs with respect to t, and $\langle L'_n \rangle$ is some enumeration of disjoint pseudolegs whose union is D (so, in particular, each L_n is a subset of exactly one L'_m), then f has the unique itinerary property with respect to L if and only if f has the unique itinerary property with respect to L'.*

Proof. ($\Leftarrow$) Obvious. ($\Rightarrow$) By contradiction. Without loss of generality, we may assume that the pseudolegs are enumerated so that if L'_n intersect the orbit of t, then $L_n \subseteq L'_n$. Suppose that f has the unique itinerary property with respect to L, but not with respect to L', and let ι and ι' be the itineraries with respect to L and L', respectively. Pick $x \neq y$ both in D such that $\iota'(x) = \iota'(y) = \alpha$. Then f^n is one-to-one on $[x, y]$ for all $n \in \omega$, and therefore all but countably many members of $[x, y]$ also have itinerary α (for t might be in $f^n[x, y]$ for some n). Since f is tentish with respect to L, $Pre_f(t)$ is arc-dense in $[x, y]$, so there is an integer n so that the minimally tentish subdendrite E of D contains at least two elements of $f^n[x, y]$, and therefore $I = f^n[x, y] \cap E$ is a nondegenerate interval. But every point in I has the same itinerary with respect to either L or L', contradicting that all but countably many members of $[x, y]$ (and therefore $f^n[x, y] = [f^n(x), f^n(y)]$) have the same itinerary. □

Definition 1.28. The tentish map f is said to be *self-similar* iff there is a partition into pseudolegs so that for each pseudoleg M, $f(M \cup \{t\}) = D$. We shall be more interested in maps which are self-similar and have no proper self-similar restrictions, and such maps will be called *critically self-similar*. It is easy to see that every self-similar map has a restriction that is critically self-similar (just take all points x whose orbits never enter the "extra" pseudoleg(s)).

A tentish dendrite map f is said to be *thickly tentish* iff every leg of D intersects the orbit of its turning point t and it is maximal with respect to that property, i.e., there is no tentish $g : E \to E$ with D a proper subset of E and $f = g|D$ such that every leg of E intersects the orbit of t.

Of the few results that have already been proved, it is easy to see that the use of pseudolegs instead of legs does not affect any of these results. For example, Corollary 1.20 is not affected since the kneading sequence does not change. The only nontrivial observation required is in Lemma 2.39, where we might now have t in the interval $(f^n(x), f^n(y))$, giving us the slightly weaker statement that all but countably many members of (x, y) have the same itinerary, which is still enough to get the result. In all results stated below, unless otherwise stated, it will be assumed that itineraries are being given with respect to some partition of $D \setminus \{t\}$ into pseudolegs, which may or may not be the same as the partition into legs.

The following easy result shows that, up to conjugacy, there is only one self-similar tentish map whose domain is a tree.

Proposition 1.29. *If $f : T \to T$ is a self-similar tentish map on a tree T, then T is an interval, and f is conjugate to the slope 2 tent map on the interval, with kneading sequence $\tau(f) = 01\bar{2}$.*

Proof. No tree other than an interval has the property that there are two or more disjoint open subsets whose closures are homeomorphic to the entire space (a property possessed by the domain of any self-similar map), so T is an interval. Furthermore, it is easy to see that there must be exactly two legs, that both endpoints must map to one of the endpoints, and that the turning point must map to the other endpoint, giving the itinerary $01\bar{2}$ for t. The fact that every such map is conjugate to the slope 2 tent map is well known. □

2. Continuous itinerary functions and the classification of tentish dendrite maps

If a map satisfies the unique itinerary property, then that obviously suggests the idea of identifying a point with its itinerary, which in turn suggests putting a corresponding topology on the appropriate set of symbol sequences. Attempts to find a simple definition for such a topology that was independent of the map used eventually led to the itinerary topology defined in this section. Any hesitation coming from the fact that the itinerary topology was non-Hausdorff was quickly relieved by the realization that this topology is an extremely useful tool which greatly simplifies a number of definitions and proofs. The lack of the Hausdorff property is also mitigated by the fact that there are many Hausdorff subspaces (in fact, there is a homeomorphic copy of the Hilbert Cube in these itinerary spaces, and therefore of all separable metric spaces), and we are generally using one of these subspaces at any given time. The fact that such infinite products of some non-Hausdorff finite spaces contain copies of spaces like the Hilbert Cube is not new. Set-Theoretic Topologists have been aware of this for some time (see [10]), and a similar non-Hausdorff topology has been defined in [11] which may turn out to be useful in computer graphics.

Before defining this topology, we prove a few more easy results about tentish maps.

Theorem 2.1. *If $f : D \to D$ is tentish with turning point t, and $\langle L_n \rangle$ is an enumeration of the legs so that $L_0 = \{t\}$, then $\tau = \tau(f)$ has the following properties*:

(1) *If $\tau_n = 0$, then $\tau_{j+n} = \tau_j$ for all $j \in \omega$.*
(2) *If α is a shift of τ, i.e., there is an $n \in \omega$ such that $\alpha_i = \tau_{n+i}$ for all $i \in \omega$, and $\alpha \neq \tau$, then there is a $j \in \omega$ such that $0 \neq \tau_j \neq \alpha_j \neq 0$.*

Proof. (1) is an immediate consequence of the simple fact that a second 0 appears in the itinerary of t if and only if t is periodic. (2) is an immediate consequence of Lemma 1.6, using the points t and $f^n(t)$. $\square$

Definition 2.2. A sequence τ of nonnegative integers will be called *acceptable* iff $\tau_0 = 0$, τ is not the constant zero sequence and τ satisfies properties (1) and (2) in the above theorem. If τ is acceptable, then we let $k(\tau)$ be the least integer k such that $\tau_k = \tau_1$, noting that (2) implies that such k must exist.

Note that if τ is preperiodic, then there are only finitely many different shifts of τ, so that checking whether or not τ is acceptable requires only a finite number of calculations in that case. Also, note that if τ satisfies the labeling convention, then $k(\tau)$ is the least k such that $\tau_k = 2$.

Example 2.3. $\tau = \overline{0122}$ has shifts $\overline{1220}$, $\overline{2201}$, and $\overline{2012}$ which are distinct from τ, each of which is different from τ on some coordinate where neither is zero, so τ is acceptable.

Example 2.4. $\tau = \overline{0121}$ has $\overline{2101}$ as its second shift, which is distinct from τ, but has the same value as τ on all coordinates for which neither is zero, so τ is not acceptable.

For sequences which are not preperiodic, acceptability is even easier to determine:

Example 2.5. If α is any sequence not containing zero which is not preperiodic, then $\tau = 0\alpha$ is acceptable, for it is clear that every shift of τ will differ from τ on some coordinate other than the initial coordinate. The absence of zeros in α is obviously needed to get the consistency property.

A more careful characterization of the property of acceptability will be given in Section 3.

From this point until the end of Section 6, we shall assume that we are dealing with itineraries in which only finitely many symbols are available, so that all itineraries have finite range, and there are finitely many pseudolegs. We still allow the possibility that there are infinitely many legs, provided that they have been partitioned to form finitely many pseudolegs. While it is possible to have minimally tentish dendrite maps f for which the range of f is infinite, it is not difficult to see that if $f : D \to D$ is a tentish map on a dendrite D, then f cannot be self-similar with respect to a partition having an infinite number of pseudolegs, because local connectedness of D would imply that the pseudolegs get arbitrarily small, so that their closures could not all map onto D without violating the continuity of f. In addition, some sequences τ having range ω can be realized by a minimally tentish map on a dendrite, while other such sequences can only be realized by dendroids which are not dendrites (the sequence 0123456789... being a simple example of this). Kneading sequences having infinite range, i.e., tentish dendrite maps in which the orbit of the turning point $\{t\}$ visits infinitely many components of $T \setminus \{t\}$, or tentish maps with an infinite number of pseudolegs (even if the kneading sequence is finite) involve a number of complications, and are more naturally discussed as a special case of forthcoming similar work on dendroids (discussed briefly in Section 7 below).

Definition 2.6. Although we are less interested in self-similar maps which are not critically self-similar, it costs virtually nothing to include that possibility in the construction. Thus, let $P_q = \{n\colon n \text{ is an integer and } 0 \leqslant n \leqslant q\}$, where $q \geqslant 2$ is some fixed integer. When there is no ambiguity, P will usually be written in place of P_q. We shall be interested in acceptable sequences τ whose range is contained in P. If the range of τ is not all of P, then the additional elements of P will be labels for legs (or pseudolegs) that do not intersect the orbit of the turning point. Put the following (non-Hausdorff) topology on P, which we call the *symbol topology* on P: All points other than 0 are isolated, and the only open set containing 0 is the entire space P. Let P^ω be the set of all sequences from P, and give

P^ω the usual product topology (also non-Hausdorff) generated by the above topology on P. If η is a finite sequence from P of length n (i.e., with domain $\{0, 1, \dots, n-1\}$), let B_η be the basic open set $\prod_{i\in\omega} U_i$, where $U_i = \{\eta_i\}$ if $i < n$ and $\eta_i \neq 0$, and $U_i = P$ if either $i \geqslant n$ or $\eta_i = 0$. We shall call this topology on P^ω the *itinerary topology* in order to distinguish it from another more common topology, namely the topology on P^ω obtained from the product topology of the discrete topology on P, which will be called the *Cantor topology* (because it is homeomorphic to a Cantor set). Whenever a word like open, closed, compact, etc. is used with respect to P^ω (or a subspace) without specifying which topology, the itinerary topology will always be the intended default. As already given above for sequences, the *shift map* on P^ω is the function $\sigma : P^\omega \to P^\omega$ defined by $\sigma(\langle a_0, a_1, a_2, \dots\rangle) = \langle a_1, a_2, a_3, \dots\rangle$. As is always the case for shift maps on product spaces of the form X^ω, σ is continuous. We shall also occasionally use the corresponding shift map on finite sequences, which sends a sequence of length n to one of length $n-1$ by eliminating the leftmost coordinate.

Note that P^ω contains all possible itineraries of points in a tentish map with q pseudolegs, along with some sequences that obviously cannot be the itinerary of any point in any such map. Indeed, given a tentish map $f : D \to D$ with q pseudolegs, the function ι defined by $\iota(x) = \iota(x, f)$ is a one-to-one function from D into P^ω.

One simple but extremely important property is that this topology makes the itinerary function continuous (in fact, it gives an alternate definition of the itinerary topology, and shows how to get the more general itinerary topologies which are described below).

Proposition 2.7. *If D is a dendrite, $t \in D$, $L_0 = \{t\}$, and L_i $(1 \leqslant i \leqslant q)$ are open sets such that D is the disjoint union of the L_i's $(0 \leqslant i \leqslant q)$, $f : D \to D$ is continuous, and $\iota(x) = \iota(f, x)$, then the itinerary topology is the strongest topology such that $\iota : D \to P^\omega$ is continuous* (*independent of the continuous function f*).

The range of X with respect to ι_f will not even be Hausdorff in general. Simple examples of such maps would include ones in which the kneading sequence is a period doubling. For example, on the interval $[1, 6]$, let $f(1) = 3$, $f(2) = 4$, $f(3) = 5$, $f(4) = 6$, $f(5) = 2$, and $f(6) = 1$, defining the function to be linear between consecutive integers. Then 4 (the turning point) has itinerary $\overline{012212}$, and every point in the interval $(3, 4)$ has itinerary $\overline{212}$, and $\overline{012212}$ and $\overline{212}$ cannot be separated in the itinerary topology. Of course, this function does not satisfy the unique itinerary property, and $\overline{012212}$ is not an acceptable sequence (since $\overline{012212}$ and $\overline{212012}$ cannot be separated).

As can be seen from the previous discussion and lemma above, the unique itinerary property will be the key property needed to get a Hausdorff range for the itinerary function, from which it will be a trivial consequence that the itinerary function will be a homeomorphism onto its range if and only if it is one-to-one. This lemma, when added to another trivial observation (that if $f^n(x) = t$, then $\sigma(\iota(f, x)) = \tau$, leading to the term "consistent" below), also gives us a natural combinatorial criterion for certain natural maximal subsets of P^ω for a σ-invariant set for which the Hausdorff property is not ruled out. We shall then want to study the subspace topology of these natural subsets, independent of any spaces that might have been embedded into them.

Definition 2.8. Let τ and α be sequences with $\tau_0 = 0$. Then α will be called *τ-consistent* if $\sigma^n(\alpha) = \tau$ whenever $\alpha_n = 0$. The sequence α will be called *τ-admissible* iff α is τ-consistent and for all $i \in \omega$ such that $\sigma^i(\alpha) \neq \tau$, there is an $n \in \omega$ such that $0 \neq \alpha_{i+n} \neq \tau_n \neq 0$.

Theorem 2.9. *Suppose that $\tau \in P^\omega$ is τ-consistent. Then the following are equivalent*:

(1) *τ is acceptable.*
(2) *τ is τ-admissible.*
(3) *Every pair of distinct points from $Orb_\sigma(\tau)$ can be separated by open sets in P^ω.*

Proof. The equivalence of the first two statements is trivial, and the equivalence of the third is an immediate consequence of the fact that two points can be separated in P just in case neither of them is 0. □

Proposition 2.10. *Suppose that $\tau \in P^\omega$ is an acceptable sequence, and let $\alpha \in P^\omega$ be τ-consistent. Then α is τ-admissible if and only if for every $n \in \omega$, $\sigma^n(\alpha)$ and τ can be separated by open sets in P^ω.*

Corollary 2.11. *If α is τ-admissible, then $\sigma(\alpha)$ is τ-admissible.*

Definition 2.12. If α and β are sequences from P, then we say that α and β *can be separated* if and only if there is an $n \in \omega$ such that $0 \neq \alpha_n \neq \beta_n \neq 0$ (i.e., they can be separated in P^ω by open sets in the itinerary topology).

Theorem 2.13. *Let τ be an acceptable sequence, and let α be a sequence such that $\sigma(\alpha)$ is τ-admissible.*

(1) *If $\sigma(\alpha) = \sigma(\tau)$, then α is τ-admissible if and only if $\alpha_0 = 0$.*
(2) *If $\sigma(\alpha) \neq \sigma(\tau)$, then α is τ-admissible if and only if $\alpha_0 \neq 0$.*

Proof. If the hypothesis of (1) holds, then α and τ cannot be separated, so the only way that α could be acceptable would be to have $\alpha = \tau$. If the hypothesis of (2) holds, then $\alpha_0 = 0$ would cause α not to be τ-consistent. If $\alpha_0 \neq 0$, then α would clearly be τ-consistent. To see that α is τ-admissible, note that by hypothesis, every shift of α can be separated from τ, so we only need to show that α can be separated from τ. Since $\sigma(\alpha) \neq \sigma(\tau)$, there is an $n > 0$ such that $\alpha_n \neq \tau_n$. We cannot have $\tau_n = 0$, because then $\sigma^n(\alpha)$ could not be separated from $\tau = \sigma^n(\tau)$, contradicting the hypothesis. Also, we cannot have $\alpha_n = 0$, since then we would have that $\tau = \sigma^n(\alpha)$ could not be separated from $\sigma^n(\tau)$, contradicting acceptability of τ. Thus, since neither α_n nor τ_n is 0, α and τ can be separated. □

Corollary 2.14. *Let τ be acceptable, β τ-admissible, $n \in \omega$, and let α be a sequence of length n such that $\alpha_i \neq 0$ for all $i < n$. Then there is a unique τ-admissible sequence γ such that $\sigma^n(\gamma) = \beta$, and for every $i < n$, either $\gamma_i = 0$ or $\gamma_i = \alpha_i$.*

Proof. A simple induction on n, using the previous theorem. □

Definition 2.15. Let $\tau \in P_q^\omega$ be acceptable, $q \geqslant 2$. Then define $D_{(q,\tau)} = \{\alpha \in P_q^\omega\colon \alpha \text{ is } \tau\text{-admissible}\}$, and give $D_{(q,\tau)}$ the topology inherited from the itinerary topology. If q is such that the range of τ is $\{0, 1, 2, \ldots, q\}$, then we define $D_\tau = D_{(q,\tau)}$. We let $\sigma_{(q,\tau)} = \sigma | D_{(q,\tau)}$ and $\sigma_\tau = \sigma | D_\tau$.

Our immediate goals will be to show that $D_{(q,\tau)}$ is a dendrite with $\sigma_{(q,\tau)} : D_{(q,\tau)} \to D_{(q,\tau)}$ self-similar (critically self-similar in the case of σ_τ), and that if $f : D \to D$ is any tentish dendrite map such that $\tau(f) = \tau$ and the pseudolegs of D are labeled by some subset of $\{1, 2, \ldots, q\}$, then the itinerary map ι_f gives a topological conjugacy between f and the restriction of σ to some closed subset of $D_{(q,\tau)}$. We start with the latter of these two results, which is extremely simple to prove.

Proposition 2.16. *For every acceptable sequence τ and every positive integer q such that $\{0, 1, 2, \ldots, q\}$ contains the range of τ, $D_{(q,\tau)}$ is a Hausdorff space. Even stronger, if α and β are distinct τ-admissible sequences, then α and β can be separated in P^ω by open sets.*

Proof. Let α and β be two distinct points of $D_{(q,\tau)}$. If $\alpha_n = 0$ for some $n \in \omega$, then $\sigma^n(\alpha) = \tau$ and $\sigma^n(\beta)$ can be separated by open sets by Proposition 2.10, and it is therefore easy to see that α and β can also be separated by open sets (the case $\beta_n = 0$ being symmetric). If α_n and β_n are different from 0 for all n, then let n be such that $\alpha_n \neq \beta_n$, and it is easy to see that α and β can be separated. □

Corollary 2.17. *Let $f : D \to D$ be a tentish dendrite map with $\tau = \tau(f)$, with pseudolegs labeled with some subset of $\{1, 2, 3, \ldots, q\}$. Then the itinerary map $\iota = \iota_f : D \to D_{(q,\tau)}$ is a homeomorphism onto its range, and ι is a topological conjugacy between f and $\sigma | range(\iota)$.*

Proof. Since $\iota : D \to P^\omega$ is one-to-one and continuous, and the range of ι is Hausdorff by the previous proposition, ι is a homeomorphism onto its range, since every continuous one-to-one map from a compact space onto a Hausdorff space is a homeomorphism. The fact that $\iota \circ f = \sigma \circ \iota$ is an immediate consequence of the definition of itinerary. □

This result tells us that a single topological space P^ω with the itinerary topology, along with a single natural map on that space (the shift map) contains conjugate copies of *every* tentish dendrite map having q or fewer pseudolegs. However, it should be pointed out that even though each individual tentish dendrite map can be embedded as a Hausdorff subspace in this manner, if $f : D \to D$ and $g : E \to E$ are two tentish maps with different kneading sequences, then the union of their embeddings will not be Hausdorff in general. To see this, note that the interval map with kneading sequence $01\bar{2}$ has a point with itinerary $\overline{112}$ and the interval map with kneading sequence $\overline{012}$ has a point (the turning point) with itinerary $\overline{012}$, yet the union of their natural embeddings is not Hausdorff, since $\overline{012}$ and $\overline{112}$ cannot be separated in P^ω.

Another benefit of representing tentish maps as a subset of P^ω is that the representation of the corresponding inverse limit spaces can then be represented as bi-infinite sequence, i.e., as subspaces of $P^{\mathbb{Z}}$. Given acceptable τ, a bi-infinite sequence can be called τ-admissible iff every tail is τ-admissible (when converted in the obvious way into a sequence with domain ω). It is then routine to check that the set $\hat{D}_{(q,\tau)}$ of bi-infinite τ-admissible sequences, with the subspace topology from $P^{\mathbb{Z}}$ and with the obvious shift map $\hat{\sigma}$ on bi-infinite sequences, is conjugate to the usual inverse limit of $D_{(q,\tau)}$ with bonding map $\sigma_{(q,\tau)}$. However, we shall not say much about inverse limit spaces in this paper. The few results here which mention them will assume some knowledge of inverse limits of continua (e.g., from [14]), but those results can be ignored without affecting the rest of the paper.

Lemma 2.18. *If τ is acceptable and α is τ-consistent, then there is exactly one τ-admissible sequence γ such that for all $n \in \omega$, either $\gamma_n = 0$ or $\gamma_n = \alpha_n$.*

Proof. Since every pair of distinct τ-admissible sequences has a coordinate n on which they are different and both nonzero, it is clear that there can be no more than one such γ. If α is τ-admissible, then we can let $\gamma = \alpha$, so assume that α is not τ-admissible. Then there is an $n \in \omega$ such that $\sigma^n(\alpha) \neq \tau$, and $\sigma^n(\alpha)$ and τ cannot be separated. Let $\delta = \sigma^n(\alpha)$. We cannot have $\delta_i = 0$ and $\tau_i \neq 0$ for any i, since otherwise $\tau = \sigma^i(\delta)$ and $\sigma^i(\tau)$ could not be separated, contradicting acceptability of τ. Thus whenever $\delta_i = 0$, τ_i is also 0. Thus, if we let β be the sequence of length n such that $\alpha = \beta\delta$, then $\beta_i \neq 0$ for all $i < n$, since $\sigma^n(\alpha) \neq \tau$. Thus, by Corollary 2.14, there is a τ-admissible sequence γ such that $\sigma^n(\gamma) = \tau$ and for all $i < n$, either $\gamma_i = 0$ or $\gamma_i = \beta_i = \alpha_i$. Then it is clear that this γ is as desired. □

Corollary 2.19. *If τ is acceptable and α is τ-consistent, then there is exactly one τ-admissible sequence γ such that every open set in P^ω containing γ also contains α.*

Definition 2.20. For any τ-consistent sequence α, define $\chi_\tau(\alpha)$ to be the unique γ satisfying the conclusion of the previous lemma.

Proposition 2.21. *χ_τ is a continuous function from the set of all τ-consistent sequences onto the set of all τ-admissible sequences.*

Proof. This is true of all functions on P^ω (or a subset) in which the only change is to change some coordinates from nonzero to zero, leaving all other coordinates unaltered. □

Theorem 2.22. *$D_{(q,\tau)}$ is a compact separable metric space.*

Proof. We first show that every sequence in $D_{(q,\tau)}$ has a convergent subsequence. Let S be any sequence in $D_{(q,\tau)}$. Then S is also a sequence in P^ω, so since P^ω is a compact metric space *in the Cantor topology*, there is a subsequence S' of S which converges *in the Cantor topology* of P^ω to some point α (which will be τ-consistent but may or may not be in $D_{(q,\tau)}$). Let $\beta = \chi_\tau(\alpha)$, and we claim that S' converges to β in the itinerary topology. Thus, let U be an itinerary neighborhood of β. Then U is also an itinerary neighborhood of α, and therefore a Cantor neighborhood of α (since the Cantor topology is finer than the itinerary topology). Thus U contains all but finitely many points of S', and therefore S' converges to β in the itinerary topology. (Recall that, in a non-Hausdorff space such as P^ω, a sequence can converge to more than one point.) Thus, since the itinerary topology has a countable basis, the itinerary topology on $D_{(q,\tau)}$ is compact. Since $D_{(q,\tau)}$ is a compact Hausdorff space with a countable basis, the Urysohn Metrization Theorem implies that $D_{(q,\tau)}$ is a separable metric space. □

Definition 2.23. Let τ be an acceptable sequence. If α and β are distinct τ-admissible sequences, we define the τ-admissible sequence $\mu_\tau(\alpha,\beta) = \mu(\alpha,\beta)$ as follows. Let n be least such that α_n and β_n are distinct and neither is 0, and define a sequence γ such that $\sigma^n(\gamma) = \tau$ (so $\gamma_n = 0$) and for $i < n$ γ_i is the larger of α_i or β_i (i.e., equal to both unless one is zero). Then $\gamma_i \neq 0$ for $i < n$ (since if $\alpha_i = \beta_i = 0$ then we could not have $\alpha_n \neq \beta_n$), so γ is τ-consistent. We then let $\mu(\alpha, \beta) = \chi_\tau(\gamma)$.

The intuition behind the above definition is as follows. We wish to show that D_τ is arcwise connected, and we will need to find an arc between any two points α and β of D_τ. The first step is to find some point "in between" them. If α and β are in different pseudolegs (i.e., are different from τ and have different first coordinates), then $\tau = \mu(\alpha, \beta)$ will be "between" α and β (i.e., will lie on the arc between α and β). If α and β are in the same pseudoleg (i.e., have the same first coordinate), then σ (if it turns out to be tentish as we hope to show) would be one-to-one on the arc between α and β, so we could apply σ (i.e., go to the next coordinate) and repeat the process, resulting in the process given above.

Proposition 2.24. *If α and β are distinct τ-admissible sequences, then $\mu(\alpha, \beta)$ is a τ-admissible sequence distinct from both α and β.*

Proof. Let α, β, γ, n be as in the previous proof. Then $\sigma^n(\alpha)$ is admissible, $\sigma^{n+1}(\alpha) \neq \sigma(\tau) = \sigma^{n+1}(\gamma)$. Thus, there is an $m > n$ such that $0 \neq \alpha_m \neq \gamma_m \neq 0$, so $\alpha \neq \mu(\alpha, \beta)$, and the same is true for β by the same argument. □

Theorem 2.25. *If τ is acceptable, then $D_{(q,\tau)}$ is arcwise connected.*

Proof. Let α and β be distinct elements of $D_{(q,\tau)}$. Let Q be the set of all dyadic rational numbers in the interval $[0, 1]$, i.e., all rationals between 0 and 1 (inclusive) of the form $n/2^m$. Define $g : [0, 1] \to D_{(q,\tau)}$ as follows. We first define g on Q by induction on the denominator, starting with $g(0) = \alpha$ and $g(1) = \beta$. Suppose $m \geqslant 0$ and that g has been defined at $n/2^i$ for all $i \leqslant m$. If $x \in Q$ is of the form $n/2^{m+1}$ for some odd n, define $g(x) = \mu(g((n - 1)/2^{m+1}), g((n + 1)/2^{m+1}))$. If $x \in [0, 1]$ is not a dyadic rational, then let $\langle x_j \rangle$ be a sequence of dyadic rationals converging to x, and let $g(x)$ be the limit of the sequence $\langle g(x_j)\rangle$, using the same argument as the compactness proof above to show that this limit exists. Then it is easy to see that g is a continuous function onto its range. □

Theorem 2.26. *If τ is acceptable, then $D_{(q,\tau)}$ is uniquely arcwise connected.*

Proof. By the previous theorem, we only need to show that $D_{(q,\tau)}$ contains no circles. Thus, suppose $C \subseteq D_{(q,\tau)}$ is a circle.

Case 1: There are $\alpha, \beta \in C$ such that $0 \neq \alpha_0 \neq \beta_0 \neq 0$. Let A and B be the two arcs such that $A \cap B = \{\alpha, \beta\}$ and $A \cup B = C$. Then $\{\gamma \in D_\tau\colon \gamma_0 = \alpha_0\}$ and $\{\gamma \in D_\tau\colon \gamma_0 \in P \setminus \{0, \alpha_0\}\}$ are two disjoint open sets whose union is $D_\tau \setminus \{\tau\}$, each of which intersects both A and B, so connectedness of A and B implies that $\tau \in A \cap B$, contradicting that A and B have no points in common other than α and β.

Case 2. There is an n such that $\alpha_0 \in \{0, n\}$ for all $\alpha \in C$. Then σ is one-to-one on C, so repeat the same argument on $\sigma(C)$. Only finitely many repetitions will be needed before Case 1 applies. □

Theorem 2.27. *If τ is acceptable, then $D_{(q,\tau)}$ is a dendrite.*

Proof. We only need to show that $D_{(q,\tau)}$ is locally connected, as the rest has already been shown. Thus, let U be an open set in $D_{(q,\tau)}$, and let α be a finite sequence such that the closure K of $B_\alpha \cap D_{(q,\tau)}$ in $D_{(q,\tau)}$ is a subset of U. Then it is easy to see that if $\beta \in K$, then for all n in the domain of α, either $\beta_n = \alpha_n$ or $\beta_n = 0$. Thus, when the arc between any two points in K is constructed as in Theorem 2.25, it is easy to check that the constructed arc stays inside K. Thus K is connected, and since U was arbitrary, $D_{(q,\tau)}$ is locally connected. □

Theorem 2.28. *$\sigma_{(q,\tau)}$ is a self-similar tentish map on $D_{(q,\tau)}$ with respect to the turning point τ and pseudolegs $L_i = \{\alpha \in D_\tau\colon \alpha_0 = i\}$.*

Proof. Clearly, σ is one-to-one on each leg, so τ is the only turning point. The L_i's are open subsets of $D_{(q,\tau)}$ whose union is $D_{(q,\tau)} \setminus \tau$, and σ is one-to-one on each L_i, so each L_i is a pseudoleg. Since $\iota_\sigma(\alpha) = \alpha$ for each $\alpha \in D_{(q,\tau)}$, the unique itinerary property is obvious. By Theorem 2.13, $\sigma(L_i \cup \{\tau\}) = D_{(q,\tau)}$, so $\sigma_{(q,\tau)}$ is self-similar. □

Theorem 2.29. *Let τ be acceptable. Then there exist unique invariant subdendrites D $(= D_\tau)$, D', and D'' of D_τ (which may or may not be distinct from each other) such that the restrictions of σ to these dendrites are, respectively, critically self-similar, thickly tentish, and minimally tentish.*

Proof. It is easily seen that $\sigma|[Orb_\sigma(\tau)]$ is the unique minimally tentish restriction, and that $\sigma|\{\alpha$: For every n, $\sigma^n(\alpha)$ is in the same *leg* as some member of the orbit of $\tau\}$ is the only thickly tentish restriction. To see that there is no proper self-similar restriction of σ_τ, suppose that E is a proper subdendrite of D_τ such that $\sigma|E$ is a tentish map on E (and therefore E contains at least the orbit of τ). Let U be a nonempty open subset of D_τ that misses E, let α be a member of U that is different from τ, and let $\beta = \sigma(\alpha)$. Then $(\sigma|E)^{-1}(\beta)$ has fewer than q elements, where $\{0, 1, 2, \ldots, q\}$ is the range of τ, and therefore $(\sigma|E)$ cannot be onto for every pseudoleg of E (since E must have at least q pseudolegs). □

Definition 2.30. Given an acceptable sequence τ, we let D'_τ and D''_τ be the subdendrites of D_τ such that the restriction of σ to those dendrites is, respectively, thickly tentish and minimally tentish (i.e., the D' and D'' of the previous corollary). We let $\sigma'_\tau = \sigma|D'_\tau$ and $\sigma''_\tau = \sigma|D''_\tau$.

Note. The obvious containments $D''_\tau \subseteq D'_\tau \subseteq D_\tau$ are not necessarily proper. For example, if $\tau = 01\bar{2}$, then $D_\tau = D'_\tau = D''_\tau$ (conjugate to the slope 2 tent map on the interval). On the other hand, the minimally, thickly, and critically self-similar maps with itinerary $\overline{012}$ are all distinct. More will be said about this in Section 4.

Corollary 2.31. *If $f : D \to D$ and $g : E \to E$ are both minimally tentish maps such that $\tau(f) = \tau(g) = \tau$, then f and g are conjugate. The same result is true if "minimally tentish" is replaced by either "thickly tentish" or "critically self-similar".*

Proposition 2.32. *let $f : D' \to D'$ be a thickly tentish dendrite map, let $f' : E \to E$ be a (not necessarily proper) restriction of f to some invariant nondegenerate dendrite E, and suppose that $h : E \to E$ is a homeomorphism such that $f' \circ h = h \circ f'$. Then h is the identity map on E (i.e., f' has no nontrivial self-conjugacies).*

Proof. Clearly, such a E must contain the orbit of τ. Since f is thickly tentish, there is no nontrivial partition of legs into pseudolegs, and therefore only one possible embedding of E into D_τ that makes f' conjugate to the corresponding restriction of σ. □

If it is possible to partition the legs into pseudolegs in a nontrivial way, then there is more than one way to do this, and it follows easily that the word "thickly" cannot be removed from the above result. A simple example would be as follows. In the Euclidean plane, let $X = \{(x, 0)\colon -2 \leqslant x \leqslant 2\}$, $Y = \{(0, y)\colon -1 \leqslant y \leqslant 1\}$, $T = X \cup Y$. Thus, let $f(-2, 0) = f(2, 0) = (-2, 0)$, $f(-1, 0) = (0, 0)$, $f(0, 0) = (1, 0)$, $f(1, 0) = (-1, 0)$, and extend f to X in the obvious linear way. For $(0, y) \in Y$, let $f(0, y) = (1 + |y|, 0)$. Then $f : T \to T$ is a tentish map with four legs. Note that $h(x, y) = (x, -y)$ is a nontrivial self-conjugacy for f, and that each leg of Y could be paired with either leg of X in a partition into pseudolegs.

Lemma 2.33. *If $\alpha, \beta \in D_{(q,\tau)}$ are such that $\sigma^n(\alpha), \sigma^n(\beta) \in D''_\tau$ for some positive integer n, then $\sigma^n[\alpha, \beta] \subseteq D''_\tau$.*

Proof. By contradiction. Suppose not. Then by shrinking the interval $[\alpha, \beta]$ if necessary, we may assume that $\sigma^n(\gamma) \notin D''_\tau$ for all $\gamma \in (\alpha, \beta)$. But then σ^n is one-to-one on (α, β), and $[\sigma^n(\alpha), \sigma^n(\beta)] \subseteq D''_\tau$, so that $(\alpha, \beta) \cup [\sigma^n(\alpha), \sigma^n(\beta)] \subseteq D''_\tau$ is a circle, a contradiction. □

Proposition 2.34. *If α is not an endpoint of $D_{(q,\tau)}$, then $\sigma^n(\alpha) \in D''_\tau$ for some positive integer n.*

Proof. Assume $\alpha \notin D''_\tau$, since otherwise we are done. Since α is not an endpoint of $D_{(q,\tau)}$, there is a point $\beta \in D_{(q,\tau)}$ distinct from α such that $\alpha \in [\beta, \tau]$. Then the unique itinerary property implies that there is a point $\delta \in [\alpha, \beta]$ such that $\sigma^n(\delta) = \tau$ for some n (pick n so that α_n and β_n are different). Then $\sigma^n(\alpha) \in \sigma^n[\tau, \delta] \subseteq D''_\tau$, by the previous lemma. □

Theorem 2.35. *If $X \subseteq D_\tau$ is nonempty, then $\sigma^{-1}(X)$ is connected if and only if X is connected and $\sigma(\tau) \in X$.*

Proof. ($\Leftarrow$) For each L' that is the closure of a leg L, $\sigma^{-1}(X) \cap L'$ is homeomorphic to X, and therefore connected since X is. Since $\sigma(\tau) \in X$, $\tau \in \sigma^{-1}(X)$, so the sets $\sigma^{-1}(X) \cap L'$ all have the point τ in common as L ranges over all legs, so that $\sigma^{-1}(X)$ will be connected.

($\Rightarrow$) Connectedness of X is obviously necessary, and any connected subset of D_τ intersecting more than one leg must also contain τ. □

Note that even if there are no nontrivial pseudolegs, the basic open sets B_α are not always connected. For example, the set $B_{0\alpha} \cap D_\tau$ will not be connected in D_τ if $\tau \notin B_\alpha$ (α finite).

Another possible approach to the theory of kneading sequences on connected topological spaces has been to allow the sets on which the itineraries are based to overlap, offering the advantage that these sets can then be closed, but traded off against the disadvantage that the itinerary of any point whose orbit visits the intersection of two of these sets will then not be uniquely defined. If we look at the difference between these two approaches in the setting of two legs, it is the difference between using $\{L_0, L_1, L_2\}$ and $\{\bar{L_1}, \bar{L_2}\}$ as the basis for defining itineraries of points. In the latter case, itineraries of points can be ambiguous, and that corresponds exactly to those places in which 0 appears in the itinerary defined in the former case. Thus, the 0 of the former case could be viewed as a "wild card" which could stand for either 1 or 2 in the latter case (see, e.g., [4]), and this often leads naturally to a quotient space of the Cantor Set $\{1, 2\}^\omega$. The itinerary topology gives a simple way of defining such quotient spaces.

Proposition 2.36. *Let τ be an acceptable sequence with range contained in $P = \{0, 1, 2, \dots, q\}$. Then the map χ_τ is a continuous function from the Cantor topology on $\{1, 2, \dots, q\}^\omega$ onto $D_{(q,\tau)}$.*

Proof. Continuity is a trivial consequence of the fact that $\{1, 2, 3, \dots, q\}$ is a discrete subset of P (since 0 is the only nonisolated point of P). Given any $\beta \in D_{(q,\tau)}$, let α be a the sequence in which all 0's of β (if any) are changed to 1's and all other members of the sequence remain unchanged. Then it is easy to see that $\chi_\tau(\alpha) = \beta$, so that $\chi_\tau|\{1, 2, \dots, q\}^\omega$ maps onto $D_{(q,\tau)}$. □

Although the results above were proven for the case of one turning point, some of the results can be extended to itineraries defined by more general partitions. The definition of the itinerary topology in such cases is straightforward.

Definition 2.37. Let X be a topological space, and let $\{L_s\colon s \in \Sigma\}$ be a partition of X as the union of disjoint subsets, where Σ is a set of symbols. The *symbol topology* on Σ with respect to the partition $\{L_s\colon s \in \Sigma\}$ is the obvious quotient topology on Σ defined by $U \subseteq \Sigma$ is open if and only if $\pi^{-1}(U)$ is open in X, where for $x \in X$, $\pi(x)$ is the unique s such that $x \in L_s$. The *itinerary topology* on Σ^ω (with respect to the same partition) is defined as the usual product topology of the symbol topology.

The following easy result is the obvious motivation for the definition.

Proposition 2.38. *Let $f : X \to X$ be continuous, and let $\iota : X \to \Sigma^\omega$ be the itinerary function with respect to the partition $\{L_s\colon s \in \Sigma\}$. Then the itinerary topology is the strongest topology with to respect to which this itinerary function is continuous* (*independent of the choice of f*).

Example 2.39. Let $f : D \to D$ be a continuous dendrite map, and let A be a finite subset of D. Let $\mathcal{S}$ be a partition of D such that $L \in \mathcal{S}$ if and only if L is either a singleton from A or a component of $D \setminus A$. (If f has a finite set of turning points, then this set is the natural choice for A.)

It is easy to show that the analogue of lemma holds: If f satisfies the unique itinerary property, and $x, y \in D$ are distinct, then there is an n such that $\iota_n(f, x)$ and $\iota_n(f, y)$ can be separated in the symbol topology by open sets, i.e., $\iota(f, x)$ and $\iota(f, y)$ can be separated by open sets in the itinerary topology. Thus, since the domain is compact and the range is Hausdorff, ι is a homeomorphism onto its range.

In this case, there is a kneading set (the set of itineraries of elements of A) instead of a kneading sequence, but there are still natural definitions of the terms acceptable (every shift of a member of the kneading set must be separated in the itinerary topology from every element of the kneading set) and admissible, leading to simple generalizations of most of the results in the first two sections, with the exception of the observation that there is no easy analogue of the labeling convention, so that statements about invariance up to conjugacy have to go the more tedious route of defining an obvious equivalence relation on kneading sets. Another drawback of the case for two or more turning points is that there is no obvious generalization of the $\star$ operation which will be defined in the next section, so that some of the results in the following sections will not generalize to this case, and those generalizations that do go through will usually be less elegant.

Another important consequence of the more general itinerary topologies is the following important result. A version of this result where τ need not be acceptable will be proven in Section 3.

Theorem 2.40. *Let X be a continuum, $f : X \to X$ continuous, let L_0 be a closed subset of X, and let $L_1, L_2, \ldots, L_q$ be open subsets of X such that $\{L_n\colon\ 0 \leqslant n \leqslant q\}$ is a partition of X. Suppose that every element of L_0 has the same itinerary τ with respect to this partition, and that this itinerary τ is acceptable. Then there is a conjugacy $\pi : X \to D_{(q,\tau)}$ (i.e., $\sigma \circ \pi = \pi \circ f$) such that $D''_\tau \subseteq \pi(X)$.*

Proof. Since X is connected, each of the open sets $L_1, \ldots, L_q$ must have a limit point in L_0, so the itinerary topology on P_q^ω for this partition is the same as for $\sigma_{(q,\tau)}$. Thus we can let $\pi = \chi_\tau \circ \iota$. The rest is easy. □

3. Hubbard trees, period *n*-tuplings, and renormalization

As mentioned above, $\tau = \overline{012212}$ is not acceptable. This sequence is just a typical "period doubling" (in this case of the period three sequence $\overline{012}$), and we would like to show that such n-tuplings, plus something similar which we call an ∞-tupling, are the only obstacles for a consistent sequence to be acceptable. In addition, every τ-consistent sequence τ can be realized as the itinerary of a (not necessarily tentish) unimodal dendrite map. For the case that the range of τ is a subset of $\{0, 1, 2\}$, this will be covered by a known result that certain kneading sequences are realized by Hubbard Trees [7] (with one very simple modification for larger ranges) for all but the ∞-tupling case discussed below. We also prove that unimodal dendrite maps which are not tentish are semiconjugate in a natural way to a tentish map with a similar kneading sequence.

Since much of this section will be about maps which are not necessarily tentish, we shall want to relax the requirement that the sequences τ under consideration are acceptable to just being τ-consistent.

Definition 3.1. Let Γ be the set of all infinite sequences α of nonnegative integers having the property that the range of α is a finite subset of ω, $\alpha_0 = 0$, and if $\alpha_n = 0$, then $\sigma^n(\alpha) = \alpha$ (i.e., Γ is the set of all sequences α which are α-consistent and have finite range). If $\alpha, \beta \in \Gamma$ such that α is periodic with period n (i.e., $n > 0$ is least such that $\alpha_n = 0$), then $\alpha \star \beta$ is defined to be the sequence γ given by $\gamma_i = \alpha_i$ if i is not a multiple of n, and $\gamma_{ni} = \beta_i$ for all $i \in \omega$. If α is not periodic, then $\alpha \star \beta$ is left undefined. Note that $\bar{0}$ is an identity for the operation, i.e., $\bar{0} \star \alpha = \alpha$, and if $\alpha \star \bar{0}$ is defined, then it is also α. A sequence $\gamma \in \Gamma$ that can be written as a $\star$-product $\alpha \star \beta$ in a nontrivial way (i.e., with neither α nor β equal to $\bar{0}$) will be called a *composite* sequence, and a sequence that cannot be factored in this way will be called *prime*. A member of Γ whose range consists of exactly 2 integers (i.e., 0 and one other positive integer) will be called *simple*. Note that the simple sequences are exactly those sequences of the form $0\bar{j}$ and $\overline{0j^n}$ for some positive integers j and n. If $\alpha \in \Gamma$ is neither simple nor the trivial $\bar{0}$ sequence, then we let $k(\alpha)$ be the least

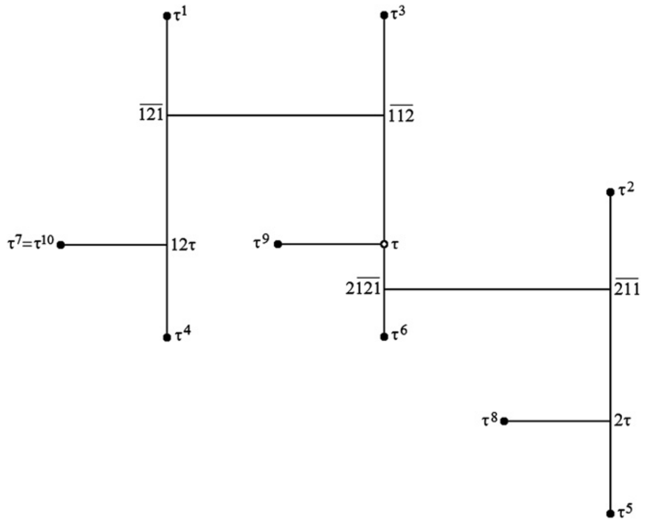

Fig. 3. The τ-minimal tree map for $\tau = \overline{012} * 012\bar{3} = 0121122\overline{123}$.

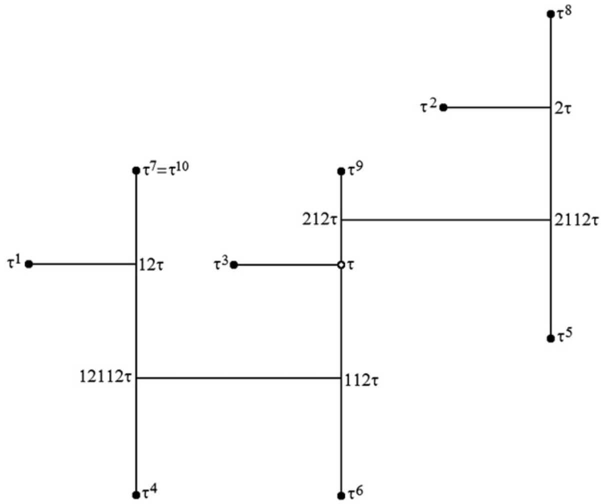

Fig. 4. The τ-minimal tree map for $\tau = 012 * 031\bar{2} = 0123121\overline{122}$.

integer $k > 1$ such that $\alpha_k \neq \alpha_1$. If $\alpha = \bar{0}$ or α is simple, we let $k(\alpha) = \infty$. If α is periodic, we let $p(\alpha)$ be the period of α, and otherwise we let $p(\alpha) = \infty$.

As pointed out above, if $\alpha \in \Gamma$, then there is a unique sequence θ such that $\alpha = \nu \circ \theta$ satisfies the labeling convention for some bijection ν of ω. Let θ^α be this unique sequence θ. Note that it will often be the case that $\theta^{\alpha\star\beta}$ and $\theta^\alpha \star \theta^\beta$ are different (because the functions corresponding to ν need not be the same for α and β). Also, it is possible for σ_β and σ_γ to be topologically conjugate, but for $\sigma_{\alpha\star\beta}$ and $\sigma_{\alpha\star\gamma}$ to be topologically distinct. See Figs. 3 and 4 for examples.

The $\star$-product is useful for analyzing both period n-tuplings and renormalization. See [8] for a simpler version defined for unimodal interval maps. It is also closely related to substitutions used for describing renormalization (see, e.g., [4]).

Proposition 3.2. *The $\star$ operation is associative.*

Theorem 3.3. $\{\tau \in \Gamma\colon\ \tau$ *is acceptable*$\}$ (*with the itinerary topology*) *contains no circles.*

Proof. By contradiction. Suppose that $C \subseteq \Gamma$ is a circle. Let n be least such that for some two elements α and β of C, $0 \neq \alpha_n \neq \beta_n \neq 0$. Let $U = \{\gamma \in P^\omega\colon\ \gamma_n = \alpha_n\}$, and $V = \{\gamma \in P^\omega\colon\ \gamma_n \notin \{0, \alpha_n\}\}$. Then U and V are disjoint open subsets of P^ω such that $\alpha \in U$ and $\beta \in V$, so there must be at least two points γ and δ of C which are in neither U nor V. Thus, $\gamma_n = \delta_n = 0$. However, γ and δ must be separated on some coordinate, and since both are periodic with period a divisor of n, there must be an $m < n$ such that $0 \neq \gamma_n \neq \delta_n \neq 0$, contradicting the choice of n. $\square$

Definition 3.4. If $\langle\alpha^0, \alpha^1, \alpha^2, \ldots\rangle$ is an infinite sequence of *periodic* elements of Γ, then it is not difficult to see that there is a natural infinite $\star$-product $\prod^\star_{i\in\omega} \alpha^i$, defined as the pointwise limit of the sequences $\beta^0 = \alpha^0$, $\beta^1 = \alpha^0 \star \alpha^1$, $\beta^2 = \alpha^0 \star \alpha^1 \star \alpha^2, \ldots$ etc. (which will exist, since the β^n's will agree on arbitrarily large initial segments as n gets large). For periodic $\alpha \in \Gamma$, we write $\alpha^{\star n}$ for the $\star$-product of α with itself n times, and $\alpha^{\star\infty}$ for the corresponding infinite product. A member of Γ that can be written as such an infinite product of nontrivial periodic members of Γ will be called *infinitely composite*. A sequence $\gamma \in \Gamma$ will be called a *tupling* of the sequence α if $\gamma = \alpha \star \beta$ for some simple β. Such a "tupling" will be called an *n-tupling* if the corresponding β has period n, and an ∞-tupling if β is of the form $0\bar{j}$.

Proposition 3.5. *If $\langle\alpha^0, \alpha^1, \alpha^2, \ldots\rangle$ is an infinite sequence of periodic elements of Γ, then $\prod^\star_{i\in\omega} \alpha^i = \alpha^0 \star \prod^\star_{i\in\omega} \alpha^{i+1}$.*

Lemma 3.6. *Let $\gamma \in \Gamma$, and suppose that $\alpha, \beta, \alpha', \beta' \in \Gamma$ with $\alpha \neq \alpha'$ both prime such that $\gamma = \alpha \star \beta = \alpha' \star \beta'$. Then α and α' are both simple with the same range, and $\alpha \star \alpha'$ is also a left factor of γ.*

Proof. Let m and n be the periods of α and α', respectively, and note that for any i which is a multiple of neither m nor n, we have $\alpha_i = (\alpha \star \beta)_i = (\alpha' \star \beta')_i = \alpha'_i$. Let $j = \gamma_1 = \alpha_1 = \alpha'_1$. Thus, if d is any common divisor of m and n, we have that $\alpha_i = \alpha'_i$ for any i that is not a multiple of d, and thus $\overline{\alpha|d} = \overline{\alpha'|d}$ is a common left factor of both α and α'. If $d \geqslant 2$, that would contradict that α and α' are distinct prime elements of Γ, so we must have that $d = 1$ is the only positive common divisor of m and n. Thus, there are integers a and b such that $am + bn = 1$. For convenience, extend α and α' to periodic bi-infinite sequences in the obvious way. By symmetry, we may assume $m < n$. Let i be an integer, $1 \leqslant i \leqslant m - 2$. Then since $\alpha_i = \alpha'_i \neq 0$ and $\alpha_{i+1} = \alpha'_{i+1} \neq 0$, we have $\alpha_i = \alpha_{i+am} = \alpha_{i+1-bn}$, and $\alpha_{i+1} = \beta_{i+1} = \beta_{i+1-bn}$. However, since α_{i+1-am} and β_{i+1-am} are both nonzero, they must be equal, and therefore $\alpha_i = \alpha_{i+1}$, i.e., $\alpha_i = j$ for all i such that $\alpha_i \neq 0$, and thus α is simple. Now suppose that i is such that $\alpha'_i \neq 0$. If i is also not a multiple of m, then $\alpha'_i = \alpha_i = j$, and if i is a multiple of m, then $i + n$ is not a multiple of m, and we have $\alpha'_i = \alpha'_{i+n} = \alpha_{i+n} = j$. Thus, the range of α' is $\{0, j\}$, and α' is also simple, with the same range as α. Since α and α' are distinct prime sequences, m and n must be distinct prime numbers, and it is then easy to check that $\overline{0j^{mn}} = \alpha \star \alpha'$ is also a right factor of γ. □

Corollary 3.7. *If $\alpha, \beta \in \Gamma$ are prime and periodic, then $\alpha \star \beta = \beta \star \alpha$ if and only if either $\alpha = \beta$ or α and β are simple with the same range.*

Corollary 3.8. *Every element of Γ can be factored into a (possibly infinite) product of prime sequences. If $\alpha \in \Gamma$ is not an ∞-tupling, then the prime factorization of α is unique up to switching the order of consecutive simple factors having the same range.*

Note that for fixed j, any infinite product of elements of the form $\overline{0j^n}$ (with the n's not necessarily fixed) is equal to $0\bar{j}$, so the requirement that α is not an ∞-tupling is necessary.

Proposition 3.9. *Let α and γ be distinct elements of Γ. Then $\gamma = \alpha \star \beta$ for some $\beta \in \Gamma$ if and only if every open set in P^ω which contains α also contains γ.*

Proof. If $\gamma = \alpha \star \beta$, then $\alpha_i = 0$ for every i such that $\alpha_i = \gamma_i$, and therefore $B_{\gamma|n} \subseteq B_{\alpha|n}$ for all n. If every open set in P^ω that contains α also contains γ, then assume $\alpha \neq \gamma$ (for otherwise we can let $\beta = \bar{0}$), and we must then have that α is periodic, and that $\gamma_i = \alpha_i$ for all i not a multiple of the period of α, i.e., α is a left factor of γ. □

The close connection between tuplings and acceptability is shown in the following theorem.

Theorem 3.10. *For every $\tau \in \Gamma \setminus \{\bar{0}\}$, τ is an acceptable sequence if and only if τ is not a tupling. In the case where τ is not acceptable, τ will be an ∞-tupling if τ is not periodic, and if τ is periodic, then τ will be an n-tupling for some n dividing the period of τ.*

Proof. ($\Rightarrow$) If $\tau = \beta \star \gamma$, where β is periodic with period n and γ is simple, then it is easy to see that τ and $\sigma^n(\alpha)$ cannot be separated, so that τ is not acceptable.

($\Leftarrow$) We prove the contrapositive, so assume that τ is not acceptable, and let $r > 0$ be least such that $\sigma^r(\tau) \neq \tau$ and $\sigma^r(\tau)$ and τ cannot be separated.

Case 1: τ is periodic with period m. Since τ is periodic and starts off with 01^i2, $1 < r < m$. Let $\alpha = \sigma^r(\tau)$ and $j = \tau_r$. Then α_i and τ_i are both different from 0 for $1 \leqslant i \leqslant m - r - 1$, and therefore since α and τ cannot be separated, $\tau_{i+r} = \alpha_i = \tau_i$ for $1 \leqslant i \leqslant m - r - 1$.

Case 2: τ is not periodic. Let $j = \tau_r$, and note that since $\tau_i \neq 0$ for all $i \geqslant 1$, $\tau_{i+r} = \tau_i$ for all $i \geqslant 1$, so that $\tau_{ir} = j$ for all $i \geqslant 1$. Let α be defined by $\alpha_{in} = 0$ for all $i \in \omega$ and $\alpha_i = \tau_i$ for all i's which are not multiples of r. Then it is easily seen that $\tau = \alpha \star 0\bar{j}$. □

This theorem gives us a number of useful corollaries that help us to easily determine whether or not a given sequence τ is acceptable.

Corollary 3.11. *If τ is a τ-consistent nonsimple periodic sequence of period a prime number, then τ is acceptable.*

Corollary 3.12. *If τ is τ-consistent but not acceptable, and τ is not periodic, then τ is an ∞-tupling of some periodic θ-consistent sequence θ.*

Corollary 3.13. *If τ is a τ-consistent sequence, and $\sigma(\tau)$ is not periodic, then τ is acceptable.*

Corollary 3.14. *If $\alpha, \beta \in \Gamma$ and α is periodic, then $\alpha \star \beta$ is acceptable if and only if β is acceptable.* (*Note that α can be simple in this corollary. Thus, a simple factor on the left does not affect acceptability as it does on the right.*)

Corollary 3.15. *If τ is an unacceptable τ-consistent sequence, then there is exactly one sequence θ which is either $\bar{0}$ or acceptable such that τ can be obtained from θ by a finite sequence of tuplings.*

Proof. Start with τ (which must be a tupling of something) and form sequences τ^n in which $\tau = \tau^0$ and τ^i is a tupling of τ^{i+1}. Since all the τ^i's are periodic with the possible exception of $\tau^0 = \tau$, the period must decrease at each step, and the process must therefore stop at either $\bar{0}$ or an acceptable sequence. While the sequence of τ^i's need not be unique, it is easy to see that the only choices are of the sort that an ab-tupling could be regarded as an a-tupling followed by a b tupling (and similarly for finite products), and that if we also assume that each τ^i is chosen to have period as small as possible, then there will be only one choice. □

The fact that every element of Γ can be realized as the kneading sequence of some unimodal dendrite map (with appropriately labeled legs) follows (with simple modifications in a couple of cases) from the proofs of known results regarding the realization of kneading sequences by Hubbard Trees. Hubbard Trees were introduced by Douady and Hubbard as a method for describing regions in the Mandelbrot Set and the dynamics on the corresponding Julia Sets (see [9]). We follow the slightly different definition of [7], which does not look at the way the tree might be embedded in the plane, or whether the embedded dynamical system might be extended to the plane in a reasonable way.

Definition 3.16. A *Hubbard Tree* is a tree T along with a function $f : T \to T$ and a distinguished point t, satisfying the following conditions:

(1) f is continuous (and surjective);
(2) every point in T has at most two preimages under f;
(3) f is locally one-to-one at all points other than t;
(4) all endpoints of T are in the orbit of t;
(5) t is periodic or preperiodic, but not fixed;
(6) if $x \neq y$ both in T are branch points or in the orbit of t, then there is an integer $n \geqslant 0$ such that $t \in f^n[x, y]$.

This definition is just a reworded version of Definition 3.1 of [7], except that in (3) we have replaced their "local homeomorphism onto its image" (at points other than t) by the simpler "locally one-to-one" (which is equivalent, since we are dealing with compact Hausdorff spaces). Note also that the word "surjective" is redundant, by (4). Also note that even though (6) is a weak version of the unique itinerary property, Hubbard Trees will not satisfy the unique itinerary property in general, and combinatorially equivalent Hubbard Trees need not be conjugate.

Proposition 3.17. *If τ is an acceptable periodic or preperiodic sequence whose range consists of* 0 *and two other integers, then D''_τ* (*equipped with the map σ''_τ and the distinguished point τ*) *is a Hubbard Tree.*

Proof. Note that (4) follows from Lemma 1.24, and that (6) is just a weak version of the unique itinerary property. The rest is trivial. □

As an immediate corollary, we get that Hubbard Trees exist for all kneading sequences which are not tuplings, giving us an alternate proof for that case. For the more general case of constructing examples of unimodal maps which realize any member of Γ as the kneading sequence, there are several possible approaches. One is to start with the examples we already have for nontuplings, and then use the observation that if one already has a unimodal example for the periodic kneading sequence α (which may be assumed to be on a tree, since the turning point is periodic), it is relatively easy to construct a unimodal example with kneading sequence $\alpha \star \beta$ for any simple β by (carefully) "blowing up" the points of the orbit of the turning point into "stars" (and also carefully blowing up all eventual preimages of these points) and defining a new function appropriately. However, the details are tedious and involve dividing into cases. Another approach which is much simpler for constructing Hubbard Trees for tuplings (other than ∞-tuplings), but is inelegant, is the following. Given a tupling α of period p, multiply α on the right by a simple prime acceptable sequence (e.g., $\beta = 01\bar{2}$), and use the results above to produce an example for the acceptable $\alpha \star \beta$. Then shrink the p small copies of D_β (with respect to σ^p) to points, and look at the result to get the combinatorial data for the desired Hubbard Tree. The other approach is to modify the proof of existence of Hubbard Trees, using what we call "voting sequences" below (see [7]). Only a trivial addition is needed to the process used in [7] (to deal with deadlocks, because t might now be a branch point).

Definition 3.18. Given $\alpha, \beta, \gamma \in D_{(q,\tau)}$, we define the *voting sequence* of the triple (α, β, γ) as a (possibly terminating) sequence of ordered triples $(\alpha^n, \beta^n, \gamma^n)$ from $D_{(q,\tau)}$ and a sequence $\delta \in P^\omega$ by induction on n as follows, starting with $(\alpha^0, \beta^0, \gamma^0) = (\alpha, \beta, \gamma)$ (and $\delta|0$ the empty sequence). Suppose that $(\alpha^n, \beta^n, \gamma^n)$ and $\delta|n$ have been defined. If α_0^n, β_0^n, and γ_0^n are all distinct, then the induction ends and we complete the sequence δ by letting $\sigma^n(\delta) = \tau$ (i.e., $\delta_i = \tau_{i-n}$ for all $i \geqslant n$). If α_0^n, β_0^n, and γ_0^n are not all distinct, then δ_n is whichever value is held by two or more of α_0^n, β_0^n, and γ_0^n. We then define $\alpha^{n+1} = \sigma(\alpha^n)$ if $\alpha_0^n = \delta_n$ (in which case we say that 'party' α 'won' the nth 'election') and $\alpha^{n+1} = \sigma(\tau)$ if $\alpha_0^n \neq \delta_n$ (in which case we say that 'party' α 'lost' the nth 'election'). β^{n+1} and γ^{n+1} are then defined in the same way. (Thus, each 'party' gets its new 'candidate' by using the shift map σ on its old 'candidate' if it 'won' the election, and using $\sigma(\tau)$ if it 'lost' the election. The induction continues until such time (if any) that the process terminates with a 'deadlock'.)

Theorem 3.19. *Let $\alpha, \beta, \gamma \in D_\tau$. Then $w(\alpha, \beta, \gamma) = \chi_\tau(\delta)$, where δ is the result of defining the voting sequence of the triple (α, β, γ) as in the previous definition.*

Proof. The proof is a simple consequence of the following three observations:

(1) If $\alpha'_0, \beta'_0, \gamma'_0$ are all distinct, then $w(\alpha', \beta', \gamma') = \tau$.
(2) If σ is one-to-one on the set $[\{\alpha', \beta', \gamma'\}]$, then $\sigma(w(\alpha', \beta', \gamma')) = w(\sigma(\alpha'), \sigma(\beta'), \sigma(\gamma'))$.
(3) If $\alpha'_0 \neq \beta'_0 = \gamma'_0$, then $w(\alpha', \beta', \gamma') = w(\tau, \beta', \gamma')$.
(4) If every pseudoleg is a leg, $\alpha'_0 \neq \beta'_0 = \gamma'_0$, and $\delta' = w(\alpha', \beta', \gamma')$, then $\delta'_0 = \beta'_0 = \gamma'_0$.

In the case where every pseudoleg is a leg, the itinerary of $w(\alpha, \beta, \gamma)$ is found one step at a time, by always making sure that the "loser" of the election is replaced by τ before applying σ to get to the next stage, so that if $[\{\alpha, \beta, \gamma\}]$ is a tree with one branching point, then $(\alpha^n, \beta^n, \gamma^n)$ will also be a tree with one branching point, with the branching points following the itinerary of $w(\alpha, \beta, \gamma) = \delta = \chi_\tau(\delta)$. In the case where there are nontrivial pseudolegs, property (4) might fail at one point of the induction (but only one time) and the resulting δ will then not be τ-admissible, but the correct itinerary can then be found by changing δ_n to 0 at the point at which (4) failed (i.e., by replacing δ by $\chi_\tau(\delta)$). $\square$

If τ is periodic or preperiodic, and $f : T \to T$ is a unimodal tree map with kneading sequence τ, where τ is not necessarily acceptable, then it is easy to see that the algorithm can still be used to find the itinerary of $w(x, y, z)$ in a finite number of steps, given the itineraries of x, y, and z, and that the result would be independent of which map f with that kneading sequence was used. Thus, τ alone is enough to give the information from which such a tree T and map f might be constructed. (See [7] for further details.)

The case of an ∞-tupling requires one additional simple observation. If $\alpha \in \Gamma$ is periodic with period n, and we want to construct a unimodal dendrite map having kneading sequence $\beta = \alpha \star 0\bar{j}$, then we can already construct a map

$f: T \to T$ whose kneading sequence is the doubling $\gamma = \alpha \star \overline{oj}$. Then T has a point $x \in [t, f^n(t)]$ whose itinerary is $\sigma^n(\beta)$, where t is the turning point of f. (Just observe that $f^n[t, f^n(t)] = [t, f^n(t)]$, and let x be the fixed point of f^n in $[t, f^n(t)]$.) It is then easy to modify f so the orbit of t converges to the orbit of x, giving an example with the desired kneading sequence.

In Section 14 of [7], Bruin and Schleicher defined generalizations for Hubbard Trees (and dendrites) in which the critical point has an infinite orbit. In this case, (5) in the definition of a "Hubbard Tree" is altered so that the Hubbard "tree" may only be a dendrite and the orbit of the critical point is not required to be finite. In addition, (4) needs to be changed to "all endpoints of the Hubbard Tree (or dendrite) are in *the closure of* the orbit of the turning point (overlooked in [7]). Using an inverse limit argument on the already constructed finite Hubbard Trees, the authors showed that infinite Hubbard Trees (and dendrites) could be constructed. In all such cases, the kneading sequences were acceptable, so that the above construction of $\sigma''_\tau : D''_\tau \to D''_\tau$ gives us an alternate (and direct) construction of the Hubbard Trees/Dendrites for the infinite case. In addition, for acceptable τ, the sets D_τ are easily seen to satisfy the definition of the "abstract Julia Sets" constructed in that section, giving an alternate method there as well.

In Theorem 2.40, we showed that maps which failed to satisfy the unique itinerary property still were semiconjugate to a tentish map, provided that the itinerary of the point used to define the partition was acceptable. This is closely related to results in Section 3 of [3], where we defined semiconjugacies that were obtained by identifying points having the same itinerary (in the case where the unique itinerary property did not hold). Such an equivalence relation is not always upper semicontinuous, so that if this relation is expanded to one that is upper semicontinuous in order to get a reasonable quotient space, then it is not always obvious whether or not the kneading sequence (or kneading set) changes, or even whether everything just collapses to a point. (For an application, see also [1].) Itinerary topologies give a more straightforward way of defining the quotient spaces defined in that paper, but we confine ourselves to the unimodal case here. Theorem 2.40 shows that we can have the kneading sequence stay the same if it is acceptable. The following theorem looks at the nonacceptable case.

Theorem 3.20. *Let X be a continuum, $f: X \to X$ continuous, let L_0 be a closed subset of X, and let $L_1, L_2, \ldots, L_q$ be open subsets of X such that $\{L_n\colon 0 \leqslant n \leqslant q\}$ is a partition of X. Suppose that every element of L_0 has the same itinerary α with respect to this partition, and that $\alpha = \tau \star \beta$ for some acceptable τ. Then there is a conjugacy $\pi: X \to D_{(q,\tau)}$ (i.e., $\sigma \circ \pi = \pi \circ f$) such that $D''_\tau \subseteq \pi(X)$. Furthermore, if f is a unimodal dendrite map, $L_0 = \{t\}$, where t is the turning point of f, then point inverses of π are connected.*

Proof. Define $\psi: P^\omega \to P^\omega$ by $\psi(\gamma\alpha) = \gamma\tau$, and $\psi(\gamma) = \gamma$ if γ is a sequence containing no zeros. Since ψ either keeps coordinates the same or changes them to zero, ψ is continuous. Thus we can let $\pi = \chi_\tau \circ \psi \circ \iota$, and as in Theorem 2.40, the rest is simple, as it is routine to prove that χ_τ, ψ, and ι all have connected point inverses if the hypothesis of the last sentence holds. □

Another property closely related to the ideas of this chapter is that of renormalizability.

Definition 3.21. A tentish map $f: D \to D$ is said to be *renormalizable* if and only if there are finitely many nondegenerate proper subcontinua $C_0, C_1, \ldots, C_{n-1}$ for some $n \geqslant 2$ (defining $C_n = C_0$) such that

(1) $f(C_i) = C_{i+1}$, $0 \leqslant i \leqslant n-1$.
(2) $C_i \cap C_j$ is either empty or a singleton if $i \neq j$.
(3) For each $i \in \{0, 1, 2, \ldots, n-1\}$, $f^n|C_i$ is a tentish map on C_i.

If the above function $f^n|C_0$ is itself renormalizable, then we say that f is *twice renormalizable*. Assuming that m-times renormalizable has been defined, we say that f is $(m+1)$*-times renormalizable* if the above $f^n|C_0$ is m-times renormalizable. Finally, f is said to be *infinitely renormalizable* if and only if it is m-time renormalizable for all positive integers m.

Proposition 3.22. *If $f: D \to D$ is a renormalizable minimally tentish map, and C_i $(0 \leqslant i \leqslant n-1)$ are as in the definition of renormalizability, then for each i, $D \setminus C_i$ has nonempty interior in D.*

Proof. Since $\bigcup_{i=0}^{n-1} C_i$ contains the orbit of the turning point, the closure $D \setminus \bigcup_{i=0}^{n-1} C_i$ consists of the union of finitely many arcs if D is minimally tentish. □

Note, however, that this result is not true in general for maps that are not minimally tentish.

Proposition 3.23. *Let $f : D \to D$ be a renormalizable tentish map, and let C_i be as in the definition of renormalizability, $0 \leqslant i \leqslant n-1$. Then exactly one of the following holds*:

(1) *The C_i's are pairwise disjoint.*
(2) $\bigcap_{i=0}^{n-1} C_i = \{c_f\}$.
(3) *There is an m, $1 < m < n$ and pairwise disjoint subcontinua C_i', $0 \leqslant i \leqslant m-1$, such that each C_i' is a connected union of C_j's, and the C_i''s also satisfy the renormalizability conditions.*

Proof. Let C_0' be the largest union of C_i's that is connected and contains C_0, and let $C_i' = f^i(C_0')$. Then there will be a positive integer m such that $C_m' = C_0'$, and exactly one of the above conditions will hold, depending on whether $m = n$, $m = 1$, or $1 < m < n$. □

Theorem 3.24. *Let $f : D \to D$ be a tentish dendrite map, and let $D'' \subseteq D$ be such that $f|D''$ is minimally tentish. Then f is renormalizable if and only if there exists an integer $n \geqslant 2$ and a nondegenerate proper subcontinuum C of D'' such that $f^n(C) \subseteq C$.*

Proof. ($\Rightarrow$) Given C_i as in the definition of renormalizability, $\bigcup_{i=0}^{n-1} C_i$ contains $Orb_f(t)$ where t is the turning point. Then let C be any one of the sets $C_i' = [C_i \cap Orb_f(t)]$.

($\Leftarrow$) Suppose that C is a nondegenerate proper subcontinuum of D such that $f^n(C) \subseteq C$, with $n \geqslant 2$. By shrinking C if necessary, we may assume that $f^n(C) = C$. Let $\mathcal{C}$ be the collection of all finite intersections of elements of $\{C, f(C), \ldots, f^{n-1}\}$. Clearly, $\mathcal{C}$ is a finite collection of subcontinua of D. Furthermore, if $X \in \mathcal{C}$, then $f(X)$ will be a subset of some element of $\mathcal{C}$, and $f^{-1}(X)$ will contain some element of $\mathcal{C}$. Thus, if we let $\mathcal{C}'$ be the set of all members of $\mathcal{C}$ having more than one point, then there will be a $C_0 \in \mathcal{C}'$ and a least integer $n' \geqslant 2$ such that $f(C_0) \subseteq C_0$, and such that C_0 is a minimal element of $\mathcal{C}'$. Then if we let $C_i = f^i(C_0)$, distinct C_i's will clearly intersect at no more than one point, and will satisfy the definition of renormalizability. □

Corollary 3.25. *A tentish map is renormalizable if and only if its minimally tentish restriction is renormalizable.*

Thus, renormalizability is a property which depends only on $\tau(f)$. The following theorem shows the connection between the $\star$ operation and renormalizability.

Theorem 3.26. *Let τ be an acceptable sequence. Then τ is composite if and only if σ_τ is renormalizable.*

Note. Additional equivalences will be proven in Section 4.

Proof. ($\Rightarrow$) Let τ be composite, say $\tau = \alpha \star \beta$, where α has period $n \geqslant 2$ and $\beta \neq \bar{0}$. Without loss of generality, we may assume α is prime. For each i, $0 \leqslant i \leqslant n-1$, let $B_i = \{\sigma^j(\tau)\colon\ j - i \text{ is a multiple of } n\}$, and let C_i be the smallest subcontinuum of D_τ containing B_i. Then $\beta \neq \bar{0}$ implies that each B_i contains at least two points, so C_i is a nondegenerate invariant subcontinuum, and $\sigma(C_i) = C_{i+1}$, $0 \leqslant i \leqslant n-1$ (where $C_n = C_0$). Note that if i and j are not multiples of n such that $j - i$ is a multiple of n, then $\tau_i = \alpha_i = \alpha_j = \tau_j$, so C_i is contained in the closure of a leg if $0 < i < n$. If I is any interval in C_0, then the tentish property implies that $f^m(I)$ contains τ as a non-endpoint for some m, which must clearly be a multiple of n. Thus, such an I cannot be contained in any C_i for $0 < i < n$, and therefore no two C_i's can intersect at more than a point. Finally, it is clear that $\sigma^n|C_0$ has no turning point other than τ (because σ is one-to-one on all of the other C_i's) and has the unique itinerary property on C_0. Since $\sigma^n|C_i$ is conjugate to $\sigma|C_0$ (again because each $\sigma|C_i$ is a homeomorphism for $0 < i < n$), we are done.

($\Leftarrow$) Suppose that σ_τ'' is renormalizable, and let C_i, $0 \leqslant i \leqslant n-1$, be as in the definition of renormalizability. Note that exactly one C_i contains τ, for otherwise every C_i would be contained in a leg and all points of C_0 not in the orbit of τ would have the same itinerary, violating the unique itinerary property, and τ cannot be in more than one C_i, for that would violate (2) in the definition of renormalizable. By renumbering if necessary, assume that $\tau \in C_0$, and for $1 \leqslant i \leqslant n-1$, let γ_i be the unique j such that $C_i \subseteq L_j$, letting $\gamma_0 = 0$ (so that γ is a sequence of length n). Let $\alpha = \overline{\gamma}$, and let β be the infinite sequence defined by $\beta_i = \tau_{in}$. Then clearly $\tau = \alpha \star \beta$. Obviously, $\alpha \neq \bar{0}$. Since C_0 is a nondegenerate continuum such that $\sigma^n(C_0) \subseteq C_0$, C_0 must contain at least two points of the orbit of τ, and therefore β must also be different from $\bar{0}$. Thus, τ is not prime. □

Corollary 3.27. *A tentish map f is infinitely renormalizable if and only if $\tau(f)$ is infinitely composite.*

4. Properties of tentish maps

We have already seen how the kneading sequence alone is enough to give us information about the dynamics of a tentish map, and in this section we wish to examine the connection between tentish maps and their kneading sequences in more detail. We show that there are a number of interesting topological properties that turn out to be equivalent to certain combinatorial properties of the corresponding kneading sequences, often with several interesting equivalences.

Many of the results of this section are made more convenient if we use the results of the previous sections to identify points with their itineraries. Thus, for the remainder of this section, if f is a tentish map on a dendrite D with q pseudolegs, we shall assume that $D \subseteq D_{(q,\tau)}$, where $\tau = \tau(f)$, and that f is the restriction of σ_τ to D. It is assumed throughout this section that τ is an acceptable sequence with range contained in $P = \{0, 1, 2, 3, \ldots, q\}$, σ is the shift map on $D_{(q,\tau)}$, and $k = k(\sigma)$. We shall also assume for convenience in this section that all acceptable sequences also satisfy the labeling convention. (The appropriate generalizations are easily obtained by replacing 1 by τ_1, 2 by τ_k, etc., in the appropriate places.)

The results of Section 2 tell us that every acceptable sequence τ has (not necessarily distinct) minimally tentish, thickly tentish, and critically self-similar tentish maps which are unique up to conjugacy, but the constructions given in that section did not tell us much about the overall structure of the dendrites or the maps which were constructed. In particular, the results proven so far do not even make it clear whether or not D_τ, D'_τ, and D''_τ are the same. We would like to have criteria for determining this and other topological properties of these dendrites, using combinatorial data from the kneading sequences whenever possible. Such results will be the most interesting when the itinerary is preperiodic, i.e., the turning point has a finite orbit. In that case, the itinerary can be described with a finite amount of information (using the notation described above), so that these results might give an algorithm for deciding specific cases. We have already seen (using voting sequences above) how the structure of the minimally tentish map can be determined if the kneading sequence is a preperiodic acceptable sequence. The following result shows that the component of $D_\tau \setminus \{\bar{1}\}$ in which a point lies is easy to determine (where $\bar{1}$ is the central fixed point).

Definition 4.1. Recall from Definition 2.3 above, that if τ is acceptable, then $k(\tau)$ is the least integer such that $\tau_k = 2$. For each sequence $\alpha \neq \bar{1}$, define k_α to be the least $n \in \omega$ such that $\alpha_n \neq 1$ ($k_{\bar{1}}$ is left undefined).

Lemma 4.2. *Let $\alpha, \beta \in D_\tau \setminus \{\bar{1}\}$. Then α and β are in the same component of $D_\tau \setminus \{\bar{1}\}$ if and only if $k_\alpha - k_\beta$ is a multiple of $k(\tau)$.*

Proof. By Theorem 1.8, $\sigma^i(\tau)$ $(0 \leqslant i \leqslant k-1)$ are in different components of $D_\tau \setminus \{\bar{1}\}$ (where $k = k(\sigma)$). For $0 \leqslant k-1$, let C_i be the component of $D_\tau \setminus \{\bar{1}\}$ containing $\sigma^{k-i}(\tau)$ and define C_i for $i \geqslant k$ by $C_i = C_{i-k}$. Note that $\sigma^k(\tau) \in C_0$. We now prove by induction on $n \in \omega$ that $\alpha \in C_{k_\alpha}$ for all $\alpha \in D_\tau$. If $k_\alpha = 0$, then clearly α is in the same component of $D_\tau \setminus \{\bar{1}\}$ as τ, i.e., C_0. Suppose that $\beta \in C_n$ for all $\beta \in D_\tau$ such that $k_\beta = n$, and let $\alpha \in D_\tau$ such that $k_\alpha = n+1$. Let m be such that $1 \leqslant m \leqslant k$ and $\sigma^m(\tau) \in C_n$. Then since $k_{\sigma(\alpha)} = n$, $\sigma(\alpha) \in C_n$ by the induction hypothesis, and therefore $\sigma(\alpha)$ and $\sigma^m(\tau) = \sigma(\sigma^{m-1}(\tau))$ are in the same component of $D_\tau \setminus \{\bar{1}\}$. Thus, since α and $\sigma^{m-1}(\tau)$ are both in $\overline{L_1}$ (on which σ is one-to-one), α and $\sigma^{m-1}(\tau)$ must be in the same component of $D_\tau \setminus \{\bar{1}\}$, i.e., C_{n+1}. □

The property of transitivity is of considerable interest in the study of dynamical systems. There are also some modifications of this property which will be of interest to us in the dendrite case.

Definition 4.3. Let X be a topological space, and let $f : X \to X$. Then f is said to be *transitive* if and only if for every pair U and V of nonempty open sets, there is a positive integer n such that $f^n(U) \cap V$ is nonempty. It is well known that if there exists an $x \in X$ such that $Orb_f(x)$ is dense in X, then f is transitive, and conversely if X is a second countable, locally compact Hausdorff space. The function f is said to have the *topological mixing property* if and only if for every nonempty open set U there is a positive integer n such that $f^n(U)$ is all of X. In addition, let us say that f is *strongly transitive* if and only if for every open set U, the set $\{x \in X\colon\ x \in f^n(U)$ for all but finitely many $n \in \omega\}$ is dense in X. It is easy to see that topological mixing implies strongly transitive, which in turn implies transitive.

We also want to define similar properties in which open sets in the above definitions are replaced by arcs. Thus, we say that a function $f : X \to X$ is *arc-transitive* iff for any two arcs A and B in X, there is a positive integer n such that $f^n(A) \cap B$ contains an arc. Similarly, we say that f is *strongly arc-transitive* if and only if for every arc A, the set $\{x \in X\colon\ x \in f^n(A)$ for all but finitely many $n \in \omega\}$ is arc-dense in X, and that f is *arc-mixing* if and only if for every arc A there is an $n \in \omega$ such that $f^n(A)$ is all of X. It is also easy to see that arc-mixing implies strongly arc-transitive which implies arc-transitive, and that for spaces in which every nonempty open set contains an arc, the arc versions of transitivity in this paragraph imply the corresponding versions in the previous paragraph. For trees, the arc-transitivity properties of this paragraph are clearly equivalent to the corresponding properties in the previous paragraph, but the following two examples will show that the properties of topological mixing and arc-transitive are independent for dendrites in general.

Example 4.4. For every integer n, define $f(n) = n + 2$ if n is even and $f(n) = n - 2$ if n is odd. Extend to the reals by defining f to be linear between consecutive integers, and then compact by letting $f(\infty) = \infty$ and $f(-\infty) = -\infty$ to get an interval map. This map is easily seen to be strongly arc-transitive but not topological mixing.

Example 4.5. Look at $\sigma : D_\tau \to D_\tau$ for any τ such that D''_τ is a tree and D_τ is not a tree (e.g., $\tau = 01\overline{12}$). Then σ_τ will be topological mixing, as proven below, but the invariant subtree D''_τ shows that σ_τ is not even arc-transitive. (Note that the complex map $f(z) = z^2 + i$ restricted to its Julia set is also an example.)

Proposition 4.6. *In a uniquely arcwise connected space, being strongly arc-transitive is equivalent to the statement that if A is an arc and x is not an endpoint of the space, then $x \in f^n(A)$ for all but finitely many $n \in \omega$.*

Proof. In one direction, the collection of non-endpoints is clearly arc-dense. In the other direction, if x is not an endpoint, then there are arcs B_1 and B_2 such that x is in every arc from a point of B_1 to a point of B_2. Assuming f is strongly arc-transitive, given an arc A, there is a positive integer n_j and a point $y_j \in B_j$ such that $y_j \in f^n(A)$ for all $n \geqslant n_j$ $(j = 1, 2)$, and $x \in f^n(A)$ for all n greater than or equal to the maximum of n_1 and n_2. □

Theorem 4.7. *If $f : D \to D$ is a strongly arc-transitive map in a uniquely arcwise connected space D, then the set of all periodic points of f is arc-dense in D.*

Proof. Let $A \subseteq D$ be an arc, and we need to show that A contains a periodic point.

Claim. *There are points $c \neq d$ both in A be such that $c \in [f^m(c), d]$ and $d \in [c, f^m(d)]$ for some positive integer m.*

Proof. Let $a, b \in A$ be non-endpoints, and by the previous proposition we know that there is a positive integer n such that $[a, b] \subseteq f^n[a, b]$. Pick $a', a'', b', b'' \in A$ such that $f^n(a') = a$, $f^n(a'') = a'$, $f^n(b') = b$, and $f^n(b'') = b'$. Then at least one of the sets $\{a', b'\}$, or $\{a'', b''\}$ will work for $\{c, d\}$, with m equal to either n or $2n$. This finishes the proof of the claim. □

Given c and d as in the claim, let x vary continuously from c to d along $[c, d]$. Then $f^m(x)$ will vary continuously from $f^m(c)$ to $f^m(d)$ (perhaps not along an arc), and there must be an $x \in [c, d]$ such that $f^m(x) = x$. □

Theorem 4.8. *If $f : D \to D$ is a strongly arc-transitive map in a uniquely arcwise connected compact space D, then the inverse limit space (D, f) is indecomposable.*

Proof. We first prove that no proper subcontinuum of $\hat{D} = (D, f)$ contains more than one periodic point of $\hat{f}$. Thus, let $\hat{C}$ be a subcontinuum of $\hat{D}$ containing two periodic points $\hat{a}$ and $\hat{b}$, and fix k such that $\hat{a}_k \neq \hat{b}_k$, let n be the least common multiple of the periods of $\hat{a}$ and $\hat{b}$, and let $a = \hat{a}_k$, $b = \hat{b}_k$. Now, suppose that $d \in D$ is not an endpoint. Then there exists a positive integer M such that $d \in f^m[a, b]$ for all $m \geqslant M$. Fix an integer j, and let i be a positive integer such that $in + j - k > M$. Then $[a, b] \subseteq \pi_{k-in}(\hat{C})$, and therefore $d \in f^{in+j-k}[a, b] \subseteq \pi_j(\hat{C})$. Thus, $\pi_j(\hat{C})$ is dense in D and is therefore all of D. Since j was arbitrary, we must have $\hat{C} = \hat{D}$. Thus, no proper subcontinuum of $\hat{D}$ contains more than one periodic point. Since $\hat{D}$ has at least three periodic points, $\hat{D}$ has at least three composants and must therefore be indecomposable. □

Lemma 4.9. *If $f : T \to T$ is a transitive tree map with finitely many turning points, then for every endpoint e there is a turning point t and a positive integer n such that $f^n(t) = e$.*

Proof. Let e be an endpoint of T. Let $e = e_0$, and define e_n by induction on n so that $f(e_{n+1}) = e_n$ and let $E = \{e_n \colon n \in \omega\}$. If e_n is not an endpoint for some n then we are done, so assume that E consists entirely of endpoints, so that E is finite. Since there are only a finite number of turning points for f, no endpoint can be a turning point, so there is an open set V containing E such that V contains no turning points and every component of V intersects E. Note that no point of $V \setminus E$ has an orbit contained entirely inside V, since otherwise the transitivity of f would be violated. Thus, some point x outside of V must map to some point of E, and x cannot be periodic, so some non-endpoint must map eventually to x, and therefore to e. If we pick n least such that t is not an endpoint and $f^n(t) = e$, then t must be a turning point. □

Theorem 4.10. *If $f : T \to T$ is a tree map with finitely many turning points, then f is strongly arc-transitive if and only if f has the topological mixing property.*

Proof. ($\Leftarrow$) Every arc has nonempty interior, so that $f^n(A) = T$ for all but finitely many $n \in \omega$.

($\Rightarrow$) Let $U \subseteq T$ be open. Then U contains an arc A, and every non-endpoint of T is in all but finitely many $f^n(A)$'s, so f is onto. However, by the previous lemma, every endpoint is an eventual preimage of some turning point (and non-endpoint), so every endpoint of T is in all but finitely many $f^n(A)$'s. Thus, some $f^n(A)$ (and therefore $f^n(U)$) contains every endpoint of T, and therefore all of T. □

Theorem 4.11. *σ_τ has the topological mixing property on D_τ, i.e., if $U \subseteq D_\tau$ is open, then $\sigma^n(U) = D_\tau$ for some n.*

Proof. Without loss of generality, U is a basic open set of the form $D_\tau \cap \prod_{i\in\omega} U_i$ where $U_i = P$ for all $i \geqslant n$. Then $\sigma^n(U) = D_\tau$. □

Corollary 4.12. *Let $\tau \neq 01\bar{2}$ be an acceptable sequence. Then the set of endpoints of D_τ is dense in D_τ.*

Proof. Since $\tau \neq 01\bar{2}$, D_τ is not a tree. Let U be an open subset of D_τ. Then $\sigma^n(U) = D_\tau$ for some n. Since D_τ has only one turning point, $\sigma^{i+1}(U)$ can have at most one endpoint more than $\sigma^i(U)$. Thus, since $\sigma^n(U)$ has infinitely many endpoints, so does U. □

Theorem 4.13. *Let τ be an acceptable sequence. Then the following are equivalent*:

(1) *τ is prime.*
(2) *$\sigma''_\tau : D''_\tau \to D''_\tau$ is not renormalizable.*
(3) *$\sigma''_\tau : D''_\tau \to D''_\tau$ is strongly arc-transitive.*
(4) *For every positive integer n, $(\sigma''_\tau)^n$ is arc-transitive.*

Proof. (1) $\Leftrightarrow$ (2) This has already been proven as the contrapositive of Theorem 3.17.

(2) $\Rightarrow$ (3) We prove the contrapositive. Suppose that σ'' is not strongly arc-transitive. Then there is an arc $A \subseteq D''_\tau$ and a non-endpoint α such that $\alpha \notin \sigma^n(A)$ for infinitely many positive integers n.

Case 1: $\sigma(\tau) \in A \subseteq [\bar{1}, \sigma(\tau)]$. Let $S = \{n\colon \sigma(\tau) \in \sigma^n(A)\}$, and let $C = \overline{\bigcup_{n\in S} \sigma^n(A)}$. Then C is a subcontinuum of D''_τ. To see that C is a proper subcontinuum of D''_τ, we show that $C \subseteq L_1 \cup \{\tau\}$. Suppose not. Then $\tau \in \sigma^n(A)$ for some $n \in S$. Fix such an n. Then $[\tau, \sigma(\tau)] \subseteq \sigma^n(A)$, so A is a subset of both $\sigma^n(A)$ and $\sigma^{n+1}(A)$, and therefore A is a subset of all but finitely many $\sigma^i(A)$ (since all but finitely many positive integers can be written in the form $an + b(n + 1)$ for some positive integers a and b). From this it easily follows that τ, and therefore every $\sigma^i(\tau)$, and therefore every non-endpoint of D''_τ, is a subset of all but finitely many $f^i(A)$'s, a contradiction. Thus C is a subset of $L_1 \cup \{\tau\}$. Let m be the least common divisor of S. Then clearly $\sigma^m(C) \subseteq C$. Since every element of $Orb_\sigma(\tau)$ is in some $\sigma^i(C)$, m cannot be 1. Thus σ'' is renormalizable by Theorem 3.24.

Case 2: Case 1 fails. Then for some i there is an arc A' that satisfies Case 1 such that $A' \subseteq \sigma^i(A)$, and the rest is easy from Case 1.

(3) $\Rightarrow$ (4) Trivial for any function, since f strongly arc-transitive implies that f^n is strongly arc-transitive (and therefore transitive) for all positive integers n.

(4) $\Rightarrow$ (2) Trivial, since if σ'' were not renormalizable, then σ''^n would be invariant on some nondegenerate proper subcontinuum $C \subseteq D''_\tau$ for some n, and any arc in C would be a counterexample to σ''^n being arc-transitive. □

Theorem 4.14. *Let τ be an acceptable sequence. Then the inverse limit space $(D''_\tau, \sigma''_\tau)$ is decomposable if and only if the leftmost prime factor of τ is simple.*

Proof. ($\Rightarrow$) If τ has a simple left factor, then σ''_τ is renormalizable, with $C_1, C_2, \ldots, C_n = C_0$ as $n \geqslant 2$ proper subcontinua of D''_τ whose union is D''_τ such that $\sigma(C_i) = C_{i+1}$ $(0 \leqslant i \leqslant n-1)$ and different C_i's intersect only at the point $\bar{1}$. Then since $(\sigma''_\tau)^{-1}(C_i) = C_{i-1}$ for $1 \leqslant i \leqslant n$, it is easy to see that the $\pi_0^{-1}(C_i)$'s (defined inside the inverse limit $(D''_\tau, \sigma''_\tau)$) are n proper subcontinua of $(D''_\tau, \sigma''_\tau)$ whose union is all of $(D''_\tau, \sigma''_\tau)$.

($\Leftarrow$) We prove the contrapositive, so assume that the leftmost prime factor of τ is not simple. The case where τ is prime is immediate from Theorem 4.8 and the previous theorem. Thus, assume that $\tau = \alpha \star \beta$, where α is a nonsimple prime. Then $\chi_\alpha : D''_\tau \to D''_\alpha$ is a continuous function such that $\chi_\alpha \circ \sigma''_\tau = \sigma''_\alpha \circ \chi_\alpha$. Thus, if γ and δ are any two periodic points of D''_τ such that $\chi_\alpha(\gamma) \neq \chi_\alpha(\delta)$, then the argument of Theorem 4.8 pulls up from D''_α to D''_τ, and the corresponding points $\hat{\gamma}$ and $\hat{\delta}$ in the inverse limit space $(D''_\tau, \sigma''_\tau)$ are in different composants. Thus $(D''_\tau, \sigma''_\tau)$ has at least three composants, and is indecomposable. □

Theorem 4.15. *Let τ be acceptable. Then the following are equivalent*:

(1) *σ''_τ is arc-transitive on D''_τ.*
(2) *σ''_τ is transitive on D''_τ.*
(3) *τ satisfies* (*at least*) *one of the following three properties*:
   (a) *τ is prime, or*
   (b) *$\tau = \alpha \star \beta$ for some α of period strictly less than $k(\tau)$ and some prime β, or*
   (c) *τ contains arbitrarily long consecutive strings of* 1*'s.*

Proof. (1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (3) We prove the contrapositive. Assuming that (a), (b), and (c) of (3) all fail, we prove that σ''_τ is not transitive. Let $k = k(\tau)$. Since (a) fails, τ has a nontrivial factorization as $\alpha \star \beta$ for some sequences α and β, with α periodic of period some positive integer $n \geqslant 2$. If more than one such factorization exists, we assume that $n > k$ if a factorization with such large n exists, and that n is as large as possible if it is impossible to get $n > k$. There are three cases:

Case 1: $n > k$. Then for each i, $0 \leqslant i \leqslant n-1$, we let $C_i = [\{\sigma^{jn+i}(\tau)\colon j \in \omega\}]$, and as in the proof of Theorem 3.21, the C_i's demonstrate the renormalizibility of σ''_τ. However, $C_k \subseteq L_2$ (since $\sigma^{jn+k}(\tau) \in L_2$ for all $j \in \omega$), and therefore $\bar{1} \notin C_i$ for all i. By Proposition 3.17, C_k has nonempty interior in D''_τ, so let $U \subseteq C_k$ be open in D_τ. Let V be an open neighborhood of $\bar{1}$ missing all C_i's. Then $\sigma^i(U) \cap V$ is empty for all i, and σ''_τ is not transitive.

Case 2: $n = k$. Then the failure of (c) implies that there is a neighborhood V of $\bar{1}$ which misses the orbit of τ. Letting C_i be as in Case 1, the rest proceeds as in Case 1.

Case 3: $n < k$. Then the fact that n is greatest possible implies that β must be prime, so that α and β demonstrate that (b) holds, contradicting that (b) fails.

(3) ⇒ (1) There are three cases, depending on which part of (3) holds.

Case 1: τ is prime. Then by Theorem 4.13, σ_τ is strongly arc-transitive and therefore arc-transitive.

Case 2: $\tau = \alpha \star \beta$ for some α of period strictly less than $k(\tau)$ and some prime β. Let $n < k_\tau$ be the period of α. Then the subcontinua $C_i = [\{\sigma^{jn+i}(\tau)\colon\ j \in \omega\}]$ are invariant under $f = \sigma^n$, and $f|C_i$ is minimally tentish on each C_i. Furthermore, $\tau(f) = \theta^\beta$, which is prime, and therefore f is arc-transitive on each C_i. Since $n < k(f)$, each C_i must hit at least two components of $D''_\tau \setminus \{\bar{1}\}$, and thus $D'' = \bigcup_{i=0}^{n-1} C_i$, and therefore σ is arc-transitive on D''_τ.

Case 3: τ contains arbitrarily long consecutive sequences of 1's. Note that this case equivalent to the statement that $\bar{1}$ is either in the orbit of τ or is a limit point of the orbit of τ. We may also assume that Case 1 fails, so that τ has a nontrivial factorization as $\alpha \star \beta$, with n as the period of τ. Since τ contains a string of at least n ones, we have $\tau_k = 2$ and $\tau_{nj+k} = 1$ for some j, and therefore the definition of $\alpha \star \beta$ implies that k is a multiple of n. Thus, we may assume that n is as large as possible, so that β is prime, and $\sigma^n|C_i = [\{\sigma^{jn+i}(\tau)\colon\ j \in \omega\}]$ is transitive by Case 1. Since $\tau \in C_i$ for all i, we are done as in Case 2. □

It has already been noted that for some itineraries, the minimally tentish maps are also maximally (and therefore also thickly) tentish, whereas for other itineraries, no tentish map has more than one of these properties. In fact, the question of whether or not a thickly tentish map is also critically self-similar turns out to involve several equivalences, some of which refer to properties of the minimally tentish map.

Theorem 4.16. *Let τ be an acceptable sequence. Then the following are equivalent*:

(1) *No tentish map with itinerary τ is both critically self-similar and thickly tentish.*
(2) *σ_τ is not thickly tentish on D_τ (i.e., $D_\tau \neq D'_\tau$).*
(3) *τ^1 is not an endpoint of D_τ.*
(4) *τ^1 is not an endpoint of D'_τ.*
(5) *For some $0 < m < n$, $\tau^n \in [\tau, \tau^m)$.*
(6) *For some $n > 0$, τ^n is not an endpoint of D''_τ.*
(7) *Either τ is periodic or there exist $0 < m < n$ such that 'party' τ^n 'wins' every 'election' in the voting sequence defined from (τ^n, τ^m, τ).*
(8) *D''_τ is a tree in which no periodic orbit of σ''_τ consists entirely of endpoints of D''_τ.*
(9) *$\sigma'\colon D'_\tau \to D'_\tau$ is not onto.*

Proof. (1) ⇔ (2) Trivial, since σ_τ is critically self-similar on D_τ.

(2) ⇒ (3) Since σ is not thickly tentish on D_τ, there must be a pseudoleg made up of two or more legs. Let α and β be two elements of such a pseudoleg that are in different legs. Then $\tau \in (\alpha, \beta)$ and σ is one-to-one on (α, β), so $\sigma(\tau) \in (\sigma(\alpha), \sigma(\beta))$ and therefore $\sigma(\tau)$ is not an endpoint of D_τ.

(3) ⇒ (4) Assuming that τ^1 is not an endpoint of D_τ, pick $\alpha \in D_\tau$ such that $\tau^1 \in (\tau, \alpha)$, and let n be least such that $\tau \in \sigma^n(\tau^1, \alpha)$. Let $\beta \in (\tau^1, \alpha)$ such that $\sigma^n(\beta) = \tau$. Then $\sigma^i[\tau^1, \beta] \subseteq D''_\tau$ for all i, and therefore $\beta \in D'_\tau$.

(4) ⇒ (3) Trivial, since $D'_\tau \subseteq D_\tau$.

(3) ⇒ (5) Assuming that τ^1 is not an endpoint of D_τ, pick $\alpha \in D_\tau$ such that $\tau^1 \in (\tau, \alpha)$, and by denseness of $Pre_\sigma(\tau)$ we may assume that $\sigma^j(\alpha) = \tau$ for some $j \in \omega$. Then σ is one-to-one on $[\tau, \alpha]$, so $\sigma^2(\tau) \in (\sigma(\tau), \sigma(\alpha))$. If σ is one-to-one on $(\sigma(\tau), \sigma(\alpha))$, then we can apply σ again and continue the induction. If not, then either $\tau \in (\sigma^2(\tau), \sigma(\alpha))$ (in which case $\sigma^2(\tau) \in (\tau, \sigma(\tau))$ and we are done) or $\tau \in \sigma(\tau), \sigma^2(\tau))$, in which case σ is one-to-one on $[\tau, \sigma(\alpha)]$ and we can continue the induction on that interval. Continuing the induction j times (if necessary) eventually gives $\sigma^n(\tau) \in (\tau, \sigma^m(\tau))$ for some distinct nonzero m, n and Lemma 1.24 implies that $m < n$.

(5) ⇔ (6) Trivial.

(5) ⇔ (7) A trivial consequence of Theorem 3.19.

(5) ⇒ (8) If $\sigma^n(\tau) \in (\tau, \sigma^m(\tau))$ for some $m < n$, and n least such that this happens, then it is easy to see that the minimally tentish restriction of σ is the dendrite (tree) with endpoints $\{\sigma(\tau), \ldots, \sigma^{n-1}(\tau)\}$, none of which has an orbit consisting entirely of endpoints.

(8) ⇒ (2) Since D''_τ is a tree, the endpoints of D''_τ must be the points τ^i, $i = 1, 2, \ldots, e$, where e is the number of endpoints, so we must have that τ^{e+1} is not an endpoint of D''_τ, for otherwise there would be a periodic orbit consisting entirely of endpoints of D''_τ. Let U be a neighborhood of τ^1 in D_τ such that $\sigma^e_\tau|U$ is one-to-one. Then τ^1 is not an

endpoint of U, and therefore τ is not an endpoint of $\sigma_t^{-1}(U) \cap L_1$, proving the existence of nontrivial pseudolegs in D_τ, so that $D_\tau \neq D'_\tau$.

(4) $\Rightarrow$ (9) Since D'_τ has no legs other than those that intersect the orbit of τ, τ^1 is an endpoint of $\sigma'(D'_\tau)$, and therefore $\sigma' : D'_\tau \to D'_\tau$ is not onto.

(9) $\Rightarrow$ (2) Trivial, since $\sigma : D_\tau \to D_\tau$ is onto. □

Theorem 4.17. *Let τ be an acceptable sequence. Then the following are equivalent.*

(1) $D''_\tau = D_\tau$.
(2) $D''_\tau = D'_\tau$.
(3) *Either $\tau = 01\bar{2}$ or $Orb_\sigma(\tau)$ is dense in D_τ.*
(4) *Either $\tau = 01\bar{2}$ or every finite sequence from $\{1, 2, 3, \dots, p\}$ appears as a subword in τ.*
(5) *σ is strongly arc-transitive on D_τ.*
(6) *σ is arc-transitive on D_τ.*
(7) *σ' is strongly arc-transitive on D'_τ.*
(8) *σ' is arc-transitive on D'_τ.*
(9) *The inverse limit of D_τ with respect to the bonding map σ_τ is an indecomposable continuum.*
(10) *The inverse limit of D'_τ with respect to the bonding map σ'_τ is an indecomposable continuum.*

Proof. (1) $\Rightarrow$ (2) Trivial from the fact that $D''_\tau \subseteq D'_\tau \subseteq D_\tau$.

(2) $\Rightarrow$ (1) By contradiction. Suppose that $D''_\tau = D'_\tau$, but $D''_\tau \neq D_\tau$. Then $D_\tau \neq D'_\tau$, so by part (8) of Theorem 4.16. D''_τ is a tree having no fixed endpoint. Note that if α is in a leg not containing any element of D_τ, then $\tau^1 \in [\alpha, \sigma(\alpha)]$, and thus that no element of $D_\tau \setminus D'_\tau$ can be a fixed point. Thus, $\bar{2}$, which is a fixed endpoint of D_τ, must also be in D'_τ, but this contradicts that $D'_\tau = D''_\tau$ has no fixed endpoints.

(1) $\Rightarrow$ (3) Suppose $D''_\tau = D_\tau$ and $\tau \neq 01\bar{2}$. Since σ''_τ is minimally tentish on D''_τ, every endpoint of D''_τ must be a limit point of the orbit of τ. Since these endpoints are dense in D_τ, the orbit of τ must also be dense in D_τ.

(3) $\Rightarrow$ (4) If α is any finite word from $\{1, 2, 3, \dots, p\}$, then B_α must intersect the orbit of τ.

(4) $\Rightarrow$ (3) This is a simple consequence of the fact that every basic open set of the form B_α contains an open set of the form $B_{\alpha'}$ where α' contains no zeros.

(3) $\Rightarrow$ (1) If $Orb_\sigma(\tau) \subseteq D''_\tau \subseteq D_\tau$ is dense in D_τ, then $D''_\tau = D_\tau$.

(1) $\Rightarrow$ (5) Given (1), (4) is also true, since that implication has already been proven. Thus, τ has arbitrarily long consecutive subsequences of 2's, and it is easy to see (given that $\tau_1 = 1$) that such a sequence must be prime. Thus, σ''_τ is strongly arc-transitive, by Theorem 4.13, and therefore so is σ_τ, since $D_\tau = D''_\tau$.

(5) $\Rightarrow$ (6) Trivial.

(6) $\Rightarrow$ (1) Since D''_τ is a closed subset of D_τ which is invariant under σ, and D''_τ contains an arc, σ_τ cannot be arc-transitive on D_τ unless D''_τ is all of D_τ.

(2) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (2) Same argument as (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1).

(5) $\Rightarrow$ (9) and (7) $\Rightarrow$ (10) Immediate from Theorem 4.8.

(9) $\Rightarrow$ (1) We prove the contrapositive. If $\alpha \in D_\tau \setminus D''_\tau$, then $\beta = (1)\alpha$ and $\gamma = (2)\alpha$ are two elements of $\alpha \in D_\tau \setminus D''_\tau$ such that $[\beta, \gamma]$ intersects D''_τ (e.g., at τ). Thus, it is easy to see that D_τ can be written as the union of two proper subcontinua B and C, both of which contain D''_τ. Then $f^{-n}(B)$ and $f^{-n}(C)$ are connected for all positive integers n (by Theorem 2.35, since they all contain $\sigma(\tau)$), and therefore $\pi_0^{-1}(B)$ and $\pi_0^{-1}(C)$ are proper subcontinua of (D_τ, σ_τ) whose union is all of (D_τ, σ_τ) (where $\pi_0 : (D_\tau, \sigma_\tau) \to D_\tau$ is the natural projection).

(10) $\Rightarrow$ (2) If $D_\tau = D'_\tau$, then this is a trivial consequence of the implications already proven, so assume in addition that $D_\tau \neq D'_\tau$. Thus, by the (2) $\Rightarrow$ (8) implication of Theorem 4.16, the fixed endpoint $\bar{2}$ is not an element of D''_τ. Let E be the smallest subcontinuum of D_τ containing D''_τ and $\bar{2}$, and note that since $\sigma(\tau)$ is an endpoint of D''_τ, it is also and endpoint of E. Then $f = \sigma|E$ is a map from E onto E. Let $\alpha \in D''_\tau \cap L_2$. Then $f|[\bar{2}, \alpha]$ is one-to-one, and therefore $\sigma(\tau) \notin f[\bar{2}, \alpha]$, so $\tau \notin [\bar{2}, \alpha]$. Thus, D'_τ contains all of E. Let G be the surjective core of σ'_τ. Then $E \subseteq G$. If E is not all of G, then G can be written as the union of two proper subcontinua B and C which both contain G, and the argument can then be done exactly as in the proof of (9) $\Rightarrow$ (1), noting that D'_τ and G have the same inverse limit with respect to σ. Thus, assume that $E = G$. Then $f^{-1}(\bar{2}) = \{\bar{2}\}$, so we can find a connected open neighborhood U

of $\bar{2}$ such that $f^{-1}(U) \subseteq U$. Then $f^{-n}(\bar{U})$ is connected for all positive integers n, and we can repeat the argument of (9) ⇒ (1) with $B = G \setminus U$ and $C = \bar{U}$. □

Corollary 4.18. *Let τ be an acceptable sequence. Then exactly one of the following holds*:

(1) $D''_\tau = D'_\tau = D_\tau$.
(2) $D''_\tau \subsetneq D'_\tau = D_\tau$.
(3) $D''_\tau \subsetneq D'_\tau \subsetneq D_\tau$.

Proposition 4.19. *Let τ be acceptable, and suppose that D''_τ is a tree with e endpoints. Then $\tau^1, \tau^2, \ldots, \tau^e$ are the endpoints of D''_τ.*

Theorem 4.20. *Let τ be acceptable, and suppose that D''_τ is a tree. Then the following are equivalent*:

(1) *D''_τ has a fixed endpoint.*
(2) *D''_τ has exactly two fixed points.*
(3) *D''_τ has more than one fixed point.*
(4) *τ is eventually constant with value different from* 1.

(*Here, "fixed" refers to the obvious function σ''_τ.*)

Proof. (1) ⇒ (2) Since $\bar{1}$ is a fixed non-endpoint, there are at least two fixed points. But any other fixed point would have to also be an endpoint, and therefore in the orbit of τ by the previous proposition. Since the orbit of τ obviously does not contain more than one fixed point, there can be no more than one fixed endpoint.

(2) ⇒ (3) Trivial.

(3) ⇒ (4) Any fixed point other than the central fixed point $\bar{1}$ must be an endpoint, and by the previous proposition, such a fixed endpoint of a tree D''_τ would have to be τ^e (where e is the number of endpoints). But τ^e is a fixed endpoint if and only if τ^e is a constant sequence with value other than 1.

(4) ⇒ (1) If τ is eventually constant with value other than 1, then some τ^n is a fixed point other than the central fixed point, and therefore a fixed endpoint. □

5. The topology of parameter space

If we look at the family f_λ of tent maps that was defined in Section 1, it is clear that they vary continuously with the parameter λ. More generally, if Y is a topological space, and for each $t \in Y$ there is a topological space X_t and a continuous function $f_t : X_t \to X_t$, then we are naturally interested in how f_t changes as t varies over different values of Y. Because of the way in which they were constructed, the spaces $D_{(q,\tau)}$ and maps $\sigma_{(q,\tau)}$ give us a very natural way of doing this for self-similar tentish maps. In order to do this, we first need to limit the acceptable sequences under consideration to avoid certain complications. One possible complication is that if we tried to do this simultaneously for acceptable sequences having arbitrarily large ranges, a sequence of acceptable sequences having finite range might limit on a sequence having infinite range, involving additional complications for acceptable sequences having infinite range which we have not yet discussed.

Definition 5.1. Fix a positive integer $q \geqslant 2$, and let $\mathcal{A}_q = \{\tau\colon\ \tau \text{ is acceptable and } \mathit{range}(\tau) \subseteq \{0, 1, 2, \ldots, q\}\}$. If we then let $\Omega_q = \{(\tau, \alpha)\colon\ \tau \in \mathcal{A}_q,\ \alpha \in D_\tau\}$, with the topology inherited from $(P^\omega)^2$, and let $\sigma^* : (P^\omega)^2 \to (P^\omega)^2$ be defined by $\sigma^*(\alpha, \beta) = (\alpha, \sigma(\beta))$, then it is clear that $\sigma^*|\Omega_q$ is a continuous function on Ω_q such that if $\tau \in \mathcal{A}_q$, then the restriction of σ^* to the set $E_{(q,\tau)} = \{\tau\} \times D_{(q,\tau)} \subseteq \Omega_q$ is conjugate to the map $\sigma_{(q,\tau)}$ on $D_{(q,\tau)}$. Thus, the very uniform way in which the $D_{(q,\tau)}$'s were constructed allows a natural continuous parameterization of the family of all of the $\sigma_{(q,\tau)}$'s with respect to the parameter τ. However, one complication is that the parameter space $\mathcal{A}_q$ of acceptable τ is not Hausdorff as a subspace of P^ω, and for that reason most of this section will concentrate on a natural subset of $\mathcal{A}_q$ which is Hausdorff in the itinerary topology. However, let us first prove a couple of basic results in which this is not necessary.

As has already been observed above, $D_{(q,\tau)}$ has a natural representation as a quotient space of Q^ω, where $Q = \{1, 2, \dots, q\}$. Thus, when we wish to study how various properties of $D_{(q,\tau)}$ change as τ varies, Q^ω gives us a convenient common domain on which to operate.

Definition 5.2. Define $\Upsilon : \mathcal{A}_q \times Q^\omega \to \Omega_q$ by $\Upsilon(\tau, \alpha) = (\tau, \chi_\tau(\alpha))$.

Theorem 5.3. *Υ is a continuous function (in the itinerary topology).*

Proof. Let $(\tau, \alpha) \in \mathcal{A} \times Q^\omega$, and let $V \subseteq \Omega_q$ be an open subset of Ω_q containing $\Upsilon(\tau, \alpha) = (\tau, \chi_\tau(\alpha))$. Without loss of generality (by shrinking V if necessary), we may assume that $V = \Omega_q \cap (B_{\tau|N} \times B_{\beta|N})$, where $\beta = \chi_\tau(\alpha)$. Since β is τ-admissible, we know that for each i such that $\beta_i \neq 0$, $\sigma^i(\beta)$ and τ can be separated, i.e., there is an integer $j(i)$ such that $0 \neq \tau_{j(i)} \neq \beta_{i+j(i)} \neq 0$. Let $M > N$ be an integer such that $i + j(i) < M$ for each $i < N$ such that $\beta_i \neq 0$. Now let $U = (B_{\tau|M} \times B_{\alpha|M}) \cap (\mathcal{A} \times Q^\omega)$. We want to show that $\Upsilon(U) \subseteq V$. Let $(\tau', \alpha') \in U$, and let $\beta = \chi'_\tau(\alpha')$. Aiming for a contradiction, suppose that $\Upsilon(\tau', \alpha') = (\tau', \beta') \notin V$, i.e., $\beta' \notin B_{\beta|N}$. For all $i < M$, whenever β_i and β'_i are both nonzero, we must have $\beta_i = \alpha_i = \alpha'_i = \beta'_i$, so since $\beta' \notin B_{\beta|N}$, there must be an $i < N$ such that $\beta'_i = 0$ and $\beta_i \neq 0$. Then $0 \neq \tau_{j(i)} \neq \beta_{i+j(i)} \neq 0$. Since $\beta'_i = 0$ and $\tau' \in B_\tau|M$, $\beta'_{i+j(i)} = \tau'_{j(i)} = \tau_{j(i)}$, and so $0 \neq \beta'_{i+j(i)} \neq \beta_{i+j(i)} \neq 0$, contradicting that $\beta'_{i+j(i)}$ and $\beta_{i+j(i)}$ can differ only if one of them is zero. □

Since the functions χ_τ map Q^ω onto $D_{(q,\tau)}$, we see from this result that $D_{(q,\tau)}$ and $\sigma_{(q,\tau)}$ vary continuously with respect to τ (identifying $D_{(q,\tau)}$ with $E_{(q,\tau)}$). A simple, although not entirely satisfactory, way of getting a metric space as the parameter space is to use the Cantor topology on the τ's instead:

Corollary 5.4. *The $D_{(q,\tau)}$'s and $\sigma_{(q,\tau)}$'s also vary continuously with respect to τ if the Cantor topology is used on acceptable sequences τ (i.e., in the embedding into $P^\omega \times P^\omega$, the Cantor topology is used on the first coordinate but we continue to use the itinerary topology on the second coordinate).*

Proof. Trivial, since the Cantor topology is stronger than the itinerary topology. □

The inelegance of this attempt is easily seen, for the Cantor topology on $\mathcal{A}_q$ has no nontrivial connected subsets, is not compact (e.g., the sequences $012(112)^n\bar{2}$ are acceptable sequences which converge in the Cantor topology to $012\overline{112}$, which is not acceptable), and every periodic member of $\mathcal{A}_q$ is an isolated point in the Cantor topology. An alternate approach that turns out to be more appealing is to limit the acceptable sequences under consideration to a subset that will be Hausdorff in the itinerary topology. The following lemma will give us a natural way of doing this.

Lemma 5.5. *If $\alpha, \beta \in \Gamma$ are not tuplings, and α and β cannot be separated, then there is a $\gamma \in \Gamma$ such that either $\beta = \alpha \star \gamma$ or $\alpha = \beta \star \gamma$, i.e., one of α and β is a left factor of the other.*

Proof. Assume α and β are not $\bar{0}$, for that case is trivial. Clearly, any two distinct nonperiodic members of Γ can be separated, so at least one of α and β is periodic. By symmetry, we may assume that α is periodic of period $n \geqslant 2$, and that if β is also periodic, then its period is at least n.

Case 1: β is not periodic, or β is periodic with some period that is a multiple of n. Then let $\gamma_i = \alpha_{ni}$, and since β_j must equal α_j for all j's that are not multiples of n, $\beta = \alpha \star \gamma$.

Case 2: β is periodic with some period m that is not a multiple of n. We show that this case cannot happen. For convenience, extend both α and β to be periodic bi-infinite sequences in the obvious way. Let p be the g.c.d. of m and n.

Subcase 2.1: $p = 1$. Then $am + bn = 1$ for some integers a and b. Let i be an integer, $1 \leqslant i \leqslant n - 2$. Then since $\alpha_i = \beta_i \neq 0$ and $\alpha_{i+1} = \beta_{i+1} \neq 0$, we have $\alpha_i = \alpha_{i+bn} = \alpha_{i+1-am}$, and $\alpha_{i+1} = \beta_{i+1} = \beta_{i+1-am}$. However, since α_{i+1-am} and β_{i+1-am} are both nonzero, they must be equal, and therefore $\alpha_i = \alpha_{i+1}$, i.e., $\alpha_i = \alpha_1$ for all i such that $\alpha_i \neq 0$, and thus α is simple (and therefore a tupling of $\bar{0}$), a contradiction.

Subcase 2.2: $p \neq 1$. Let a and b be such that $p = am + bn$, and thus if i is not a multiple of p, $\alpha_i = \alpha_{i+bn} = \beta_{i+bn} = \beta_{i+am+bn} = \beta_{i+p} = \alpha_{i+p}$ and if we let $\alpha'_i = \alpha_{ip}$, and define δ to be of period p with $\delta_i = \alpha_i$ for $0 \leqslant i \leqslant p - 1$, then $\alpha = \delta \star \alpha'$, and similarly $\beta = \delta \star \beta'$. But then Subcase 2.1 will hold for α' and β', giving the same contradiction. □

The following definition is motivated directly by the above lemma.

Definition 5.6. We define a sequence $\alpha \in \Gamma$ to be *pseudoprime* if and only if there does not exist a nontrivial factorization $\alpha = \beta \star \gamma$ such that β is acceptable. Let $\mathcal{P}_q = \{\tau \in \mathcal{A}_q\colon \tau \text{ is pseudoprime}\}$. We let $\mathcal{Z} = \{\alpha \in \Gamma\colon \mathcal{Z} \text{ is infinitely composite}\}$, and $\mathcal{Z}_q = \mathcal{Z} \cap \mathcal{P}_q$.

Proposition 5.7. *A sequence $\alpha \in \Gamma$ is pseudoprime if and only if one of the following two mutually exclusive conditions holds.*

(1) *α factors into finitely many prime sequences such that the rightmost factor is not simple and all other factors are simple.*
(2) *α factors as the product of infinitely many simple sequences, but is not an ∞-tupling* (*i.e., there is not a tail in the factorization which all have the same domain*).

Proof. This is a simple consequence of Lemma 5.5 plus the fact that α is acceptable if and only if α is not a tupling. $\square$

Corollary 5.8. *If $\gamma \in \Gamma$ is not a product of simple sequences, then there exists a unique pseudoprime $\alpha \in \Gamma$ such that $\gamma = \alpha \star \beta$ for some $\beta \in \Gamma$.*

Note. We consider $\bar{0}$ to be the empty product of simple sequences.

Definition 5.9. As it turns out, for reasons that will not be apparent until later, it is the sequences satisfying the first of the properties of Proposition 5.4 that will be of the greatest interest to us. Thus, let us define a sequence $\gamma \in \Gamma$ to be *uniform* if and only if it can be written in the form $\alpha \star \beta$, where α is the (possibly empty) product of finitely many simple elements of Γ, and $\beta \in \Gamma$ is prime. The above α will be unique and periodic for any given uniform γ, and we define $s(\gamma)$ to be the period of that α (with $s(\gamma) = 1$ if γ is prime). Define a function $\chi' : \Gamma \to \Gamma$ by letting $\chi'(\alpha)$ be the (unique) uniform left factor of α if there is one, and $\bar{0}$ otherwise. Let us define a member α of Γ to be *semisimple* if every prime factor of α is simple (i.e., if $\chi'(\alpha) = \bar{0}$). Obviously, every uniform sequence is pseudoprime, and if α has a nonsimple prime factor, then $\chi'(\alpha)$ is uniform. Let $\mathcal{L}_q = \{\tau \in \mathcal{P}_q\colon \tau \text{ is uniform}\}$, noting that $\mathcal{L}_q = \mathcal{P}_q \setminus \mathcal{Z}_q$. If $\alpha \in \mathcal{L}_q$, then the unique semisimple β and prime γ such that $\alpha = \beta \star \gamma$ will be called the *semisimple part* and the *prime part* of α, respectively.

In Section 6, we shall show that a tentish dendrite map with finitely many pseudolegs is tentlike if and only if its kneading sequence is uniform, thus giving a simple characterization of which tentish maps are tentlike. For the remainder of this section, we wish to establish some of the main properties of the itinerary topology on $\mathcal{P}_q$ and $\mathcal{L}_q$ (especially the latter), showing in particular that the one point compactification $\mathcal{M}_q$ of $\mathcal{L}_q$ is a dendrite. Although the arguments are often reminiscent of those used in Section 2, the details are significantly more difficult, due to the fact that we are not working with a single acceptable sequence during most of the proofs here.

Theorem 5.10. *$\mathcal{P}_q$ is Hausdorff, and is maximal in the sense that no larger set of acceptable sequences is Hausdorff in the itinerary topology.*

Proof. An immediate consequence of the previous proposition, since acceptable sequences are not tuplings. $\square$

Definition 5.11. For each $\alpha \in \Gamma$, we define $N(\alpha)$ to be the integer N satisfying the following two properties, if such an N exists.

(1) There is a $\beta \in B_{\alpha|N} \cap \Gamma$ and a semisimple left factor γ of β which is not a left factor of α.
(2) For every $\beta \in B_{\alpha|N+1} \cap \Gamma$, and for every semisimple left factor γ of β, γ is also a left factor of α.

If no such N exists, then we define $N(\alpha)$ to be ∞. If $N(\alpha) < \infty$, we define $A_\alpha = B_{\alpha|N(\alpha)+1}$.

Note that since the neighborhoods $B_{\alpha|N}$ shrink as N gets larger, there can be no more than one N satisfying both (1) and (2). We could also have defined $N(\alpha)$ to be the greatest integer satisfying (1) or the least integer satisfying (2). The numbers $N(\alpha)$ are useful in investigating the topology of $\mathcal{L}_q$ because it will be possible to avoid certain complications when working inside neighborhoods of the form $B_{\alpha|n}$, where $n > N(\alpha)$, i.e., where condition (2) holds. The strategy will be to piece together the structure of itinerary topology on $\mathcal{P}_q$ and $\mathcal{L}_q$ by first looking at these sets, A_α in particular.

Proposition 5.12. *If α is semisimple, then $N(\alpha) = \infty$.*

Proof. If α is periodic and semisimple, then pick any simple β, and $\alpha \star \beta \in B_{\alpha|N}$ for all N. Suppose α is not semisimple and not periodic, i.e., α is the infinite product of simple sequences. Given N, let β be a left factor of α of period greater than N. Then $\beta \star \overline{01}$ and $\beta \star \overline{02}$ are both in $B_{\alpha|N}$, but cannot both be left factors of α. □

Proposition 5.13. *If α is a left factor of β, then $N(\beta) \leqslant N(\alpha)$.*

Proof. Trivial, since $B_{\beta|N} \subseteq B_{\alpha|N}$ for all N in that case. □

Proposition 5.14. *For all $\alpha \in \Gamma$, $N(\alpha) \geqslant 3$.*

Proof. Let $i = \alpha_1$, $j = \alpha_2$. Then $B_{\alpha|3}$ contains both $\overline{0i} \star \overline{0j} \star \overline{01}$ and $\overline{0i} \star \overline{0j} \star \overline{02}$, and α cannot have both of these as a left factor. □

Theorem 5.15. *If $\alpha \in \mathcal{L}_q$ is prime, then $N(\alpha) < \infty$ and is equal to the smallest N such that the set $\{n \leqslant N\colon \alpha_n \notin \{0, \alpha_1\}\}$ has* g.c.d. 1.

Proof. Let $j = \alpha_1$. Then any prime simple factor of an element of $B_{\alpha|N}$ ($N \geqslant 2$) would have to be of the form $\overline{0j^{p-1}}$ for some prime p. If $S(N) = \{n \leqslant N\colon \alpha_n \notin \{0, \alpha_1\}\}$ has greatest common divisor 1, and p is prime, then some element of S is not divisible by p, so that no element of $B_{\alpha|N+1}$ could have $\overline{0j^{p-1}}$ as a left factor, and thus no element of $B_{\alpha|N+1}$ has any semisimple left factor. On the other hand, if N is least such that $S(N)$ has g.c.d. 1, and p is a prime factor of $\{n < N\colon \alpha_n \notin \{0, \alpha_1\}\}$, then define β by $\beta_{pi} = \alpha_{pi}$ and $\beta_i = j$ for all i's not divisible by p. Then $\beta \in B_{\alpha|N}$ and $\overline{0j^{p-1}}$ is a left factor of β but not of α. □

Corollary 5.16. *If $\alpha \in \mathcal{L}_q$ is prime and periodic, then $N(\alpha)$ is either less than $p(\alpha)$ or equal to $p(\alpha) + k(\alpha)$*

Proof. Let $N = N(\alpha)$, $p = p(\alpha)$, $k = k(\alpha)$, Suppose that $N \geqslant p$. Since α_N must be different from 0 and α_1, we must have $N \geqslant p + k$. Let $S = \{n \leqslant p + k\colon \alpha_n \notin \{0, \alpha_1\}\}$. Then $k, p + k \in S$, so $p \in Add(S)$ and therefore $\{n \in \omega\colon \alpha_n \notin \{0, \alpha_1\}\} \subseteq Add(S)$, so $Add(S) = \mathbb{Z}$, and S has g.c.d. 1. □

Note that $N(\overline{0122}) = 3$ and $N(\overline{012}) = 5$, so that both possibilities mentioned in the corollary can occur.

Corollary 5.17. *If $\alpha \in \mathcal{L}_q$ is prime, then $N(\alpha) < 2p(\alpha)$.*

Lemma 5.18. *If α is simple and periodic, $\beta \in \mathcal{L}_q$, $N(\beta) < \infty$, and either $\beta_1 \neq \alpha_1$ or β is prime, then $N(\alpha \star \beta) = p(\alpha)N(\beta)$.*

Proof. Let $p = p(\alpha)$, $N = N(\beta)$, $j = \alpha_1$. Then $B_{\beta|N}$ has an element γ which has a semisimple left factor (say δ) which is not a left factor of β. Then $\alpha \star \delta$ is a semisimple left factor of $\alpha \star \gamma \in B_{\alpha\star\beta|pN}$ which is not a left factor of $\alpha \star \beta$. Thus $N(\alpha \star \beta) \geqslant pN$.

For the other direction, let $\gamma \in B_{\alpha\star\beta|pN+1}$, and let δ be a semisimple left factor of γ. If δ has α as a left factor, say $\delta = \alpha \star \delta'$ and $\gamma = \alpha \star \gamma'$, then $\gamma' \in B_{\beta|N+1}$, so δ' is a left factor of β (by definition of $N(\beta)$), and therefore δ is a left factor of $\alpha \star \beta$. Thus, the only case left is where δ is simple. Since $\delta_1 = \alpha_1$, $range(\delta) = \{0, j\}$.

Case 1: $\alpha_1 \neq \beta_1$. Then the period of δ must divide $k(\alpha \star \beta) = p(\alpha)$, so δ is a left factor of α, and therefore of α.

Case 2: $\alpha_1 = \beta_1$. Then β is prime, by the hypothesis of the lemma. Thus, 1 is the g.c.d. of $\{n \leqslant N\colon \beta_n \notin \{0, j\}\}$, and therefore p is the g.c.d. of $S = \{n \leqslant pN\colon \alpha \star \beta_n \notin \{0, j\}\}$. Thus, since $S \subseteq S' = \{n \leqslant pN\colon \gamma_n \notin \{0, j\}\}$, the period of δ is a divisor of the g.c.d. of S' and therefore of p, so δ is a left factor of α.

Thus, δ is a left factor of $\alpha \star \beta$ in all cases, and we are done. □

Theorem 5.19. *$N(\alpha) < \infty$ for all $\alpha \in \mathcal{L}_q$, and if α is semisimple and $\beta \in \mathcal{L}_q$, then $N(\alpha \star \beta) = p(\alpha)N(\beta)$.*

Proof. Every uniform sequence α can be written as the product of a finite number of simple sequences on the left and one nonsimple prime sequence on the right in such a way that adjacent factors in the product have different ranges (by multiplying together any adjacent simple sequences that happen to have the same range). It is then easy to calculate $N(\alpha)$ using the previous lemma and induction, from which the result easily follows. □

Corollary 5.20. *$\mathcal{Z}_q$ is a closed subset of $\mathcal{P}_q$.*

Proof. For $\alpha \in \mathcal{L}_q$, the neighborhoods A_α clearly do not contain any infinitely composite sequences. □

Corollary 5.21. *If X is a compact subset of $\mathcal{L}_q$, then $\{s(\alpha)\colon \alpha \in X\}$ is bounded.*

Theorem 5.22. *Let $\alpha \in \mathcal{L}_q$. Then for every $n > N(\alpha)$ and every γ in the closure in Γ of $U = B_{\alpha|n} \cap \mathcal{L}_q$, if $\delta \in \Gamma$ is a finite product of simple sequences which is a left factor of γ, then δ is also a left factor of α.*

Proof. This is true of all elements of U by the definition of $N(\alpha)$, so we only need to check members of the boundary of U. Thus, suppose that $\gamma \in \Gamma$ is in the boundary of U, and let δ be a semisimple left factor of γ. Since γ is in the boundary of U, we must have $\gamma_i = 0$ for at least one $i < n$ such that $\alpha_i \neq 0$. Since α and γ cannot be separated on any coordinate less than n, it is easy to find a nonperiodic η such that $\gamma \star \eta \in U$. Then δ is a left factor of $\gamma \star \eta$, so since $n > N(\alpha)$, δ is also a left factor of α. □

It is easy to see that $\mathcal{L}_q$ is not compact, for the sequences $\overline{01^n} \star \overline{012}$ are all in $\mathcal{L}_q$, but their only limit point in Γ is easily seen to be $0\bar{1}$. On the other hand, we will be able to show that $\mathcal{L}_q$ is locally compact and locally a dendrite. We avoid the complication of semisimple sets by working inside the sets $A_\alpha = B_{\alpha|N(\alpha)+1}$.

Theorem 5.23. *For every $\alpha \in \mathcal{L}_q$, the closure of $A_\alpha \cap \mathcal{L}_q$ in $\mathcal{L}_q$ is compact.*

Proof. Let $N = N(\alpha)$, $U = B_{\alpha|N}$. It is sufficient to prove that every sequence from the closure (in $\mathcal{L}_q$) of U has a subsequence which converges to some element of $\mathcal{L}_q$. Given such a sequence S, it has a subsequence S' which converges in the Cantor topology, therefore also in the weaker topology of P^ω. It is also easy to see that the limit (call it β) must be a member of Γ. Since β is clearly in the closure *in Γ* of U, Theorem 5.22 implies that every semisimple left factor of β is also a left factor of α. Thus, β cannot be semisimple, so $\chi'(\beta)$ will be the desired limit in $\mathcal{L}_q$. □

Corollary 5.24. *$\mathcal{L}_q$ is locally compact metric space in the itinerary topology.*

Definition 5.25. Define $\mathcal{M}_q$ to be the one point compactification of $\mathcal{L}_q$, with o as the additional point. Since $\mathcal{M}_q$ has a countable basis, $\mathcal{M}_q$ is a compact metric space. Define $\chi''\colon \Gamma \to \mathcal{M}_q$ by letting $\chi''(\alpha)$ be the unique uniform left factor β of α is such β exists and letting $\chi''(\alpha)$ be o otherwise (i.e., the same as χ' with $\bar{0}$ replaced by o).

The following result shows why the restriction to points of $\mathcal{P}_q$ that were not infinitely composite was necessary.

Proposition 5.26. *$\mathcal{P}_q$ is not locally compact at any point of $\mathcal{Z}_q$.*

Proof. Given $\alpha \in \mathcal{Z}_q$ and $U = B_{\alpha|N}$, let β be a left factor of α such that $\beta \in U$ (easily done since α has left factors of arbitrarily large period). Let $\gamma^{(n)} = \beta \star \overline{01^n} \star \overline{0212}^{\star\infty}$. Then $\gamma^{(n)} \in U \cap \mathcal{Z}_q$, but their limit is the ∞-tupling $\beta \star 0\bar{1}$. □

Proposition 5.27. *Let $\alpha \in \Gamma$ be a finite product of periodic simple sequences. Then the map $h_\alpha : \mathcal{M}_q \to \mathcal{M}_q$ defined by $h(o) = o$ and $h(\beta) = \alpha \star \beta$ for $\beta \neq o$ is a homeomorphism onto its range.*

Theorem 5.28. *For every $\alpha \in \mathcal{L}_q$, and for every $n > N(\alpha)$ the closure of $B_{\alpha|n} \cap \mathcal{L}_q$ in $\mathcal{L}_q$ is arcwise connected.*

Proof. Similar to the proof of Theorem 2.15. Let K be the closure in $\mathcal{L}_q$ of $B_{\alpha|n} \cap \mathcal{L}_q$. Given $\beta, \gamma \in U$, we define a map $g : Q \to K$ on the set Q of dyadic rationals of the form $i/2^m$ by induction on the denominator. Let $g(0) = \beta$ and $g(1) = \gamma$. If $g(i/2^j)$ has been defined for all $j \leqslant m$, let $x = i/2^{m+1} \in Q$ for some odd i, and let $\beta' = g(i - 1/2^{m+1})$ and $\gamma' = g(i + 1/2^{m+1})$. If $\beta' = \gamma'$, let $g(x) = \beta' = \gamma'$. Otherwise, let r be the first coordinate on which β' and γ' can be separated, and note that we cannot have $\beta_i = \gamma_i = 0$ for any $i < r$ because then β' and γ' could not be separated. Thus, let δ be the element of Γ of period r defined by letting $\delta_i = \max\{\beta_i', \gamma_i'\}$. Note that δ will be in the closure of $B_{\alpha|n} \cap \Gamma$ in Γ, so that $g(x) = \chi'(\delta)$ will be in K. Extend g to $[0, 1]$ by taking limits at members of $[0, 1] \setminus Q$ (again using χ' when needed to make sure the range is in K). □

Corollary 5.29. *For every $\alpha \in \mathcal{L}_q$, and for every $n > N(\alpha)$ the closure of $B_{\alpha|n} \cap \mathcal{L}_q$ in $\mathcal{L}_q$ is a local dendrite.*

Proof. Local connectedness is an immediate consequence of the fact that we proved the previous theorem for arbitrarily small neighborhoods. It has already been shown that Γ contains no circles. □

Having seen that the closure of each $A_\alpha = B_{\alpha|N(\alpha)+1}$ is a dendrite, we now want to see how these pieces fit together to form the spaces $\mathcal{P}_q$, $\mathcal{L}_q$, and $\mathcal{M}_q$. The key to this is determining what kind of boundary points a given A_α can have in $\mathcal{L}_q$.

Theorem 5.30. *Let $\alpha \in \mathcal{L}_q$ be prime and suppose that $N(\alpha) < p(\alpha)$. Then A_α has either one or two boundary points in $\mathcal{L}_q$. The point $\beta = \overline{\alpha|N(\alpha)}$ will always be such a boundary point such that $p(\beta) = N(\alpha)$ and $N(\beta) = N(\alpha) + k(\alpha)$. The only other possible boundary point in $\mathcal{L}_q$ (if there is one) will be one point γ such that $p(\gamma) < N(\alpha) = N(\gamma)$.*

Proof. Let $N = N(\alpha)$, $j = \alpha_1$. Since $\alpha_i \neq 0$ for all $i \leqslant N$, the only possible boundary points of A_α are of the form $\overline{\alpha|n}$ for some $n \leqslant N$. The cases where $n \leqslant k(\alpha)$ can be ruled out since they are all simple and therefore not in $\mathcal{L}_q$. Given any such δ on the boundary, $\delta_i \notin \{0, j\}$ implies $\alpha_i \notin \{o, j\}$, so we must have $N(\delta) \geqslant N(\alpha)$. For the case $n = N$, it is clear that $\beta = \overline{\alpha|N(\alpha)}$ is a boundary point, as is $\beta' = \chi'(\beta) \in \mathcal{L}_q$, but the latter cannot be different from β, since $N(\alpha) \leqslant N(\gamma') < 2p(\gamma')$. Thus, β is one boundary point with period N and $N(\beta) = N + k(\beta) = N + k(\alpha)$. Suppose that $\gamma = \overline{\alpha|n}$ is also a boundary point of A_α in $\mathcal{L}_q$, where $n < N$. Then $p(\gamma) = n < N \leqslant N(\gamma)$, so since in this case $N(\gamma)$ is the least $i > p(\gamma)$ such that $\gamma_i \notin \{o, j\}$, and $\gamma_N \notin \{o, j\}$, we must have $N(\alpha) = N(\gamma) = p(\gamma) + k(\gamma) = n + k(\alpha)$, so $n = N(\alpha) - k(\alpha)$, and there is only one such possible n □

Note that the second boundary point will happen just in case $\overline{\alpha|n}$ happens to be a boundary point of A_α for $n = N(\alpha) - k(\alpha)$, i.e., if $\alpha_i = \alpha_{i-n}$ for $n + 1 \leqslant i \leqslant N$. Examples of both cases are easy to find. For example, if $\alpha = \overline{012112}$ (with $p(\alpha) = 6$, $N(\alpha) = 5$), then $\beta = \overline{01211}$ (with $p(\beta) = 5$, $N(\beta) = 7$) and $\gamma = \overline{012}$ (with $p(\gamma) = 3$, $N(\gamma) = 5$) are both boundary points of A_α in $\mathcal{L}_2$. If $\alpha = \overline{01122}$ (with $p(\alpha) = 5$, $N(\alpha) = 4$), then $\beta = \overline{0112}$ (with $p(\beta) = 4$, $N(\beta) = 7$) is the only boundary point of A_α in $\mathcal{L}_2$.

Theorem 5.31. *Let $\alpha \in \mathcal{L}_q$ be prime and suppose that $N(\alpha) = p(\alpha) + k(\alpha)$. Then A_α has at least q boundary points in $\mathcal{L}_q$. Exactly one of these points (say β) will have $p(\beta) = N(\alpha)$ and $N(\beta) = N(\alpha) + k(\alpha)$, and all such remaining boundary points γ will have $N(\gamma) < N(\alpha)$.*

Proof. Let $p = p(\alpha)$, $N = N(\alpha)$, $k = k(\alpha)$, $j = \alpha_1$. This theorem is more difficult than the previous one, because a boundary point can now have many values at p (provided that its period is larger than p). It is easy to see that any boundary point of A_α must be of the form $\delta = \overline{\eta}$, where η is a sequence of length $n \leqslant N$ obtained by letting $\eta_i = \alpha_i$ for $i < n$ and $i \neq p$ and then letting η_p be any one of $1, 2, \ldots, q$ (the latter only if $n > p$). Of these possibilities, most will not be boundary points of A_α (because the part after n does not match appropriately with α). In addition, the case $n = p$ can be ruled out (because it just gives us α). We look at the possibilities case by case.

Case 1: $n = N$. In this case, we clearly get q boundary points of A_α in Γ (but perhaps not in $\mathcal{L}_q$). Applying χ' to these q points will give us q boundary points in $\mathcal{L}_q$.

Subcase 1a: $\delta_p = j$. Then there are no new coordinates at N or below which are different from 0 or j, so $N(\delta) \geqslant N(\alpha)$. However, N was least such that $\{i \leqslant N\colon \alpha_i \notin \{0, j\}\}$ had g.c.d. 1, so since $\delta_N = 0$, $\{i \leqslant N\colon \delta_i \notin \{0, j\}\}$ has g.c.d. greater than 1, $N(\delta) > N$ and therefore $N(\delta) = N + k$ (since $k(\delta) = k$). The fact that $\chi'(\delta) = \delta$ (so that $\delta \in \mathcal{L}_q$) is the same argument as before: δ cannot have period less than half of $N(\delta)$.

Subcase 1b: $\delta_p \neq j$. Then $S_1 = \{i \leqslant p\colon \delta_i \notin \{0, j\}\}$ is obtained by removing N and adding p from $S_2 = \{i \leqslant N\colon \alpha_i \notin \{0, j\}\}$, so S_1 and S_2 have the same g.c.d., since $N = p + k$ and $k \in S_1 \cap S_2$. Thus, $N(\delta) = N(\chi'(\delta)) = p < N$.

Case 2: $n < N$. The case where n divides N, if relevant at all, was covered in Subcase 1b (by the application of χ'). Thus, we may assume that $\delta_N = \alpha_N$, so that $N(\delta) \leqslant n + k \leqslant N$. But $n + k = N$ is impossible, since that would give us $n = p$, so $N(\delta) < N(\alpha)$. □

The sequence $\alpha = \overline{012}$ (with $p(\alpha) = 3$, $N(\alpha) = 5$) is an example where A_α has only the q boundary points $\overline{012i1}$ of period 5, with $N(\overline{01211}) = 7$ and $N(\overline{012i1}) = 3$ for the values of i other than 1. The sequence $\alpha = \overline{01211}$ (with $p(\alpha) = 5$, $N(\alpha) = 7$) is an example where A_α has q boundary points $\overline{01211i1}$ of period 7, with $N(\overline{0121111}) = 9$ and $N(\overline{01211i1}) = 5$ for the values of i other than 1, and also another boundary point $\overline{012}$

Theorem 5.32. *Let $\alpha \in \mathcal{L}_q$, Then there is exactly one boundary point $\beta \in \mathcal{L}_q$ of A_α such that $s(\beta) = s(\alpha)$ and $N(\beta) > N(\alpha)$. If $\gamma \in \mathcal{L}_q$ is a boundary point of A_α other than β, then at least one of the following occurs*:

(1) $s(\gamma) < s(\alpha)$;
(2) $N(\gamma) < N(\alpha)$;
(3) $p(\gamma) < p(\alpha)$.

Proof. Let $\gamma \in \mathcal{L}_q$ be a boundary point of A_α such that none of (1), (2), (3) occur. We need to show that there is only one such γ. Since $s(\gamma) = s(\alpha)$ ($s(\gamma) > s(\alpha)$ being impossible by the definition of $N(\alpha)$), we can find a semisimple η of period $s = s(\alpha)$ and prime α' and γ' with $\alpha = \eta \star \alpha'$ and $\gamma = \eta \star \gamma'$ (with Theorem 5.22 guaranteeing that it is the same η in both cases). Then $sN\gamma' = N(\gamma) \geqslant N(\alpha) = sN(\alpha')$, so $N(\gamma') \geqslant N(\alpha')$, and similarly, $p(\gamma') \geqslant p(\gamma)$. Then, by Theorems 5.24 and 5.25, there can be only one such γ', and we have $N(\gamma') > N(\alpha')$ (since the case $N(\gamma') = N(\alpha')$ would give $p(\gamma') < p(\alpha')$, and thus $p(\gamma) < p(\alpha)$, a contradiction. Thus there is only one boundary point β not satisfying either (1), (2), or (3), and it satisfies $N(\beta) > N(\alpha)$. □

This theorem motivates the following definition.

Definition 5.33. If $\alpha \in \mathcal{L}_q$, define the *rank* of α, written $\rho(\alpha)$, to be the ordered triple $\langle s(\alpha), N(\alpha), p(\alpha)\rangle$. Among all such ordered triples $\langle s, N, p\rangle$, let $<$ be the lexicographical ordering, and observe that $<$ is a well-ordering on the set $\omega\times(\omega \cup \{\infty\})(\times(\omega \cup \{\infty\})$.

Corollary 5.34. *If $\alpha \in \mathcal{L}_q$, then there is exactly one boundary point $\beta \in \mathcal{L}_q$ of A_α such that $\rho(\alpha) < \rho(\beta)$, and for all other boundary points $\gamma \in \mathcal{L}_q$ of A_α, we have $\rho(\gamma) < \rho(\alpha)$.*

Theorem 5.35. *$\mathcal{M}_q$ contains no circles.*

Proof. Since Γ contains no circles, neither does $\mathcal{L}_q$, so any circle in $\mathcal{M}_q$ would have to contain o. Suppose C is such a circle. Let $\alpha \in C \setminus \{o\}$ have least possible rank (using the fact that $<$ is a well ordering on triples). Then $A_\alpha \cap \mathcal{L}_q$ is an open set in $\mathcal{M}_q$ containing some, but missing many, points of C. Thus, the boundary of A_α must contain at least two points of C. However, all but one of these boundary points have rank strictly less than C, contradicting the choice of α. □

Definition 5.36. For each $\alpha \in \mathcal{L}_q$, let $\alpha^\sharp$ be the unique element of the boundary of A_α in $\mathcal{L}_q$ such that $\rho(\alpha) < \rho(\alpha^\sharp)$. Define by induction $\alpha^{\sharp n+1} = (\alpha^{\sharp n})^\sharp$.

Proposition 5.37. *For all $\alpha \in \mathcal{L}_q$, $s(\alpha^\sharp) = s(\alpha)$, and $k(\alpha^\sharp) = k(\alpha)$.*

Proposition 5.38. *If $\alpha \in \mathcal{L}_q$ and $\alpha_1 = j$, then the sets $\{i < p(\alpha)\colon \alpha_i \notin \{0, j\}\}$ and $\{i < p(\alpha^\sharp)\colon \alpha^\sharp_i \notin \{0, j\}\}$ have the same* g.c.d.

Proposition 5.39. *If $p = p(\alpha) < N(\alpha) =$ and α is prime, then $\alpha^\sharp_i = \alpha_1$ for all i such that $p \leqslant i < N$.*

Proposition 5.40. *If β is semisimple and α is prime, $(\beta \star \alpha)^\sharp = \beta \star (\alpha^\sharp)$.*

Proposition 5.41. *If $\alpha \in \mathcal{L}_q$, then α and $\alpha^\sharp$ have the same semisimple left factors.*

Lemma 5.42. *For every $\alpha \in \mathcal{L}_q$, the sequence $\langle \alpha^{\sharp n} \rangle$ converges to a sequence $\beta \in \mathcal{L}_q$ such that $s(\beta) > s(\alpha)$.*

Proof. *Case* 1: α is prime. Let $j = \alpha_1$. Since $\alpha^{\sharp n}$ and $\alpha^{\sharp n+1}$ agree below $p(\alpha^{\sharp n})$, and $p(\alpha^{\sharp n})$ is increasing, it is clear that the $\alpha^{\sharp n}$'s converge in Γ to some sequence β. We need to see that β satisfies the specified conditions. Note that Proposition 5.39 implies that β has a final sequence of j's. Since all of the sets $\{i < p(\alpha^{\sharp n})\colon \alpha^{\sharp n}_i \notin \{0, j\}\}$ have the same g.c.d., the g.c.d. of the set $\{i\colon \beta_i \notin \{0, j\}\}$ is the same, and is greater than 1, but a divisor of $k(\beta) = k(\alpha^{\sharp n})$. Thus, β is not prime, and has a simple left factor η with $p(\eta)$ a divisor of $k(\alpha)$. If $\beta = \eta \star \beta'$, we wish to show that β is prime. Note that the set $\{i\colon \beta'_i \notin \{0, j\}\}$ has g.c.d. equal to 1, so that if $\beta'_1 = j$, then we are done. Thus, suppose that $\beta'_1 \neq j$, then β is prime, since it has a final sequence of j's. Thus $\beta \in \mathcal{L}_q$ and $s(\beta) > s(\alpha)$.

Case 2: α is not prime. The $\alpha = \eta \star \alpha'$ for some semisimple η and prime α', and Case 1 can be applied to α'. □

Definition 5.43. Given $\alpha \in \mathcal{L}_q$, define $\alpha^{\sharp\infty}$ to be the β of the conclusion of the previous lemma.

Lemma 5.44. *No component of $\mathcal{L}_q$ is compact.*

Proof. Suppose X is a compact connected subset of $\mathcal{L}_q$. Then X is covered by finitely many set of the form A_α, so $\{s(\alpha)\colon \alpha \in X\}$ is bounded. Thus, let $s = s(\alpha)$ for some $\alpha \in X$ be such that $s(\beta) \leqslant s$ for all $\beta \in X$. Then there is an arc I_n from $\alpha^{\sharp n}$ to $\alpha^{\sharp n+1}$ in $\mathcal{L}_q$, so $X \cup \bigcup_{n\in\omega} I_n$ is connected, and therefore so is $X \cup \bigcup_{n\in\omega} I_n \cup \alpha^{\sharp\infty}$. But $\alpha^{\sharp\infty} \notin X$, so X is not a maximal connected subset. □

Corollary 5.45. *$\mathcal{M}_q$ is arcwise connected, and therefore a dendrite.*

Proof. Given $\alpha \in \mathcal{L}_q$, the component of $\mathcal{L}_q$ containing α is not compact, and therefore has o as a limit point, so it is routine to find an arc from o to α. □

Theorem 5.46. *$\mathcal{Z}_q$ is zero-dimensional in the itinerary topology.*

Proof. If $\alpha \in \mathcal{Z}_q$, then $\alpha_i \neq 0$ for all $i > 0$, so for every $\alpha \in \mathcal{Z}_q$ and every $N > 1$, no boundary point of $B_{\alpha|N}$ is in $\mathcal{Z}_q$. □

Theorem 5.47. *Each component of $\mathcal{P}_q$ is a dendrite, and contains exactly one element of $\mathcal{Z}_q$.*

Proof. Since $\mathcal{M}_q$ is a dendrite, a component C of $\mathcal{L}_q$ will be a maximal arcwise connected subset. Such a set cannot have more than one limit point in $\mathcal{Z}_q$, since that would give us a circle in $\mathcal{M}_q$. If $\alpha \in C \cap \mathcal{L}_q$, then let $\alpha^{(0)} = \alpha$ and $\alpha^{(n+1)} = (\alpha^{(n)})^{\sharp\infty}$. Then it is easy to check that the $\alpha^{(n)}$'s are in the same component of $\mathcal{L}_q$ and that they converge to a member β of $\mathcal{Z}_q$. Thus, $C \cup \{\beta\}$ is a dendrite. □

Note that if $1 \leqslant i < j \leqslant q$, and $n \geqslant 1$, then $B_{0i^n j}$ is a set in which every boundary point in Γ is simple, and that the sets $B_{0i^n j} \cap \mathcal{L}_q$ are clopen in $\mathcal{L}_q$, so that $\mathcal{L}_q$ has infinitely many components. However, it cannot have uncountably many components, so uncountably many components of $\mathcal{P}_q$ are singletons. The following simple results allow us to characterize which components of $\mathcal{P}_q$ are singletons.

Proposition 5.48. *If α is prime, then $N(\alpha^{\sharp\infty}) \leqslant N(\alpha) - 1 + 2s(\alpha^{\sharp\infty})$.*

Proof. Let $\alpha' = \alpha^{\sharp\infty}$, $N = N(\alpha)$, and $j = \alpha_1$. Then, as in Lemma 5.34, $\alpha'_i = j$ for all $i \geqslant N$. Let β be semisimple and γ prime such that $\alpha' = \beta \star \gamma$, and let N' be largest such that $\gamma_{N'} \neq j$. Then by Theorem 5.10, $N(\gamma) \leqslant N'$ if $\gamma_1 = j$ and $N(\gamma) \leqslant N' + 2$ if $\gamma_1 \neq j$, so $N(\gamma) \leqslant N' + 2$ in either case, and thus $N(\alpha') = s(\alpha)N(\gamma) \leqslant N's(\alpha') + 2s(\alpha')$. Since clearly $N's(\alpha') \leqslant N - 1$, the result follows. □

Corollary 5.49. *If $\alpha = \beta \star \gamma$ and $\alpha^{\sharp\infty} = \beta \star \beta' \star \gamma'$, where β is semisimple, β' is simple, and γ and γ' are prime, then $N(\gamma') \leqslant ((N(\gamma) - 1)/p(\beta')) + 2 \leqslant N(\gamma)$, with equality if and only if $N(\gamma) = 3$.*

Corollary 5.50. *If $\alpha \in \mathcal{L}_q$, then finitely many iterations of the $(.)^{\sharp\infty}$ operation on α produces an element of Γ whose prime part is of the form $0i\bar{j}$, after which future iterations of the $(.)^{\sharp\infty}$ operation give elements whose prime parts alternate between $0i\bar{j}$ and $0j\bar{i}$ (where $i \neq j$).*

Proof. Let $\alpha^{(0)} \in \mathcal{L}_q$ and define $\alpha^{(n+1)} = (\alpha^{(n)})^{\sharp\infty}$, and let $\gamma^{(n)}$ be the prime part of $\alpha^{(n)}$. The previous corollary tells us that $N(\gamma^{(n+1)}) < N(\gamma^{(n)})$ unless $N(\gamma^{(n)}) = 3$, in which case $N(\gamma^{(n+1)})$ is also 3. Since $N(\gamma) \geqslant 3$ for all $\gamma \in \mathcal{L}_q$, we must reach a point where $N(\gamma^{(n)}) = 3$. If γ is prime and $N(\gamma) = 3$, then $\gamma^{\sharp\infty} = 0ij\bar{i} = \overline{0i} \star 0j\bar{i}$, where $i = \gamma_1$ and $j = \gamma_2$. □

Corollary 5.51. *If $\alpha \in \mathcal{Z}_q$, then the component of $\mathcal{P}_q$ containing α is nondegenerate if and only if α is of the form $\beta \star (\overline{0i} \star \overline{0j})^{\star\infty}$ for some $i \neq j$.*

Proof. The ($\Leftarrow$) direction is immediate from the previous corollary. For the ($\Rightarrow$) direction, note that the limit of the iteration of the $(.)^{\sharp\infty}$ operation on $\beta \star 0i\bar{j}$ is $\beta \star (\overline{0i} \star \overline{0j})^{\star\infty}$ (for semisimple β). □

Definition 5.52. Let $S = \{z \in \mathbb{C}\colon |z| = 1\}$, and let $f : S \to S$ be defined by $f(z) = z^2$. For each point $a \in S \setminus \{1\}$, let $\{L_0^a, L_1^a, L_2^a\}$ be the partition of S in which L_0^a contains the two complex square roots of a, L_1^a is the component of $S \setminus L_0^a$ containing a, and L_2^a is the other component of $S \setminus L_0^a$. Note that both members of L_0^a have the same itinerary, which we call the kneading sequence τ^a. We let $\iota^a(z)$ be the itinerary of the point $z \in S$ with respect to the partition $\{L_0^a, L_1^a, L_2^a\}$. For convenience, we let $\tau^1 = \bar{0}$, but we leave ι^1 undefined. Note that the function stays the same while the partition varies. In the literature (e.g., [4,5,7]), one usually finds the symbols $\star$, 1, 0 used for 0, 1, 2, respectively.

Lemma 5.53. *The map $h : S \to \mathcal{M}_2$ defined by $h(a) = \chi''(\tau^a)$ is continuous.*

Proof. Let $a \in S$, and let U be a neighborhood of $h(a)$.

Case 1: $h(a) = \alpha \neq o$. Then we may assume (by shrinking U if necessary) that $U = B_{\alpha|n}$ for some $n > N(\alpha)$. Let $E = \{m < n\colon \tau_m^a \neq 0$, and $V = \bigcap_{m\in E} f^{-1}(L_{\tau_m^a})$. Then V is an open subset of S such that $a \in V$ and $h(V) \subseteq U$.

Case 2: $h(a) = o$, $a \neq 1$. Then $\mathcal{M}_2 \setminus U$ is compact, and can therefore be covered by finitely many sets of the form $B_\gamma|N\gamma$. Pick a positive integer N that is strictly larger than all the relevant N_γ's. Then the set V defined as in Case 1 contains a and has $h(V) \subseteq U$.

Case 3: $a = 1$. Given the $B_{\gamma|N_\gamma}$'s and N as in Case 2, find a neighborhood V of 1 such that for all $z \in V \setminus \{1\}$, $\tau^z|N = 01^{N-2}$. Then V will be as desired. □

Assuming that the Mandelbrot Set is locally connected, this leads to a connection between $\mathcal{M}_2$ and the Mandelbrot Set which follows easily from results proven above added to known results about the Mandelbrot Set.

Theorem 5.54. *Assume that the Mandelbrot Set M is locally connected, and let ∂M be the boundary of M. Then there is a continuous function $g : \partial M \to \mathcal{M}_2$ such that whenever $c \in \partial M$ is such that the Julia set J_c of the function $z^2 + c$ is locally connected and $g(c) \neq o$, then the restriction of $z^2 + c$ to J_c is semiconjugate to $\sigma_\tau : D_\tau \to D_\tau$, where $\tau = g(c)$, with conjugacy in the case where the Julia Set map is tentish with a uniform kneading sequence.*

Proof. It is well known (see, e.g., [4,5,7]) that if the Mandelbrot Set is locally connected, then there is a continuous $F : \partial M \to S$ such that if $F(c_1) = F(c_2)$, then the corresponding kneading sequences are the same. Thus, the map $g(c) = h(F^{-1}(c))$ gives a well defined continuous map from M into $\mathcal{M}_2$. Furthermore, if J_c is locally connected, then there is a semiconjugacy F_c from S onto J_c having the property that if $F_c(z_1) = F_c(z_2)$, then z_1 and z_2 have itineraries α_1 and α_2 which are *usually* the same with respect to the partition $\{L_0^{F(c)}, L_1^{F(c)}, L_2^{F(c)}\}$, and at worst have the property that $\chi_{g(c)}(\alpha_1) = \chi_{g(c)}(\alpha_2)$, (see, e.g., [5], pp. 383–385, from which this statement easily follows). Thus, the remainder of the result follows easily from Theorem 3.20. □

6. The characterization of tentlike maps

In this section, we consider the problem of whether or not a tentish map $f : D \to D$ with turning point t is tentlike, i.e., whether or not there exists a taxicab metric and a positive constant $\lambda > 1$ such that f multiplies distances by λ on all components of the complement of $D \setminus \{t\}$. Although it is clear that all tentlike maps are tentish, it is not immediately obvious that the converse is not true. The periodic examples of tentish not tentlike maps having smallest period all have period 9 kneading sequences of the form $\tau = \overline{0abcabdab}$ (where $a \neq b$ and $c \neq d$), and even in these cases, the proof that σ''_τ is not tentlike involves some work. In this section, we prove that a tentish map f with finitely many pseudolegs is tentlike if and only if $\tau(f)$ is uniform. We also prove that expansion factors λ_τ are unique for tentlike maps, that the corresponding taxicab metric d_τ is unique up to scale, and that there is a way of choosing this d_τ so that the D_τ's, σ_τ's, λ_τ's, and d_τ's all vary continuously with respect to the parameter τ for all uniform τ.

It is well known that if $f : [0, 1] \to [0, 1]$ is a piecewise linear map of the unit interval such that each linear piece has slope either λ or $-\lambda$, where $\lambda \geqslant 1$, then the topological entropy $h(f)$ of f is $\log\lambda$. It is also known that if f is a map on the unit interval with finitely many turning points, then $h(f) = \lim_{n\to\infty} \frac{1}{n} N(f^n)$, where $N(g)$ is the number of turning points of g. Thus, for tent maps of the interval (and for tentlike maps on trees as well), the expansion factor of the tent(like) map has a close relationship to the topological entropy of the map. From this, we can observe that a tentish map on an interval having kneading sequence $(\overline{01} \star \overline{02})^{\star\infty}$ (e.g., at the "Feigenbaum limit") has topological entropy zero, and is therefore not tentlike. However, in the case of dendrites which are not trees, there is not necessarily a connection between the expansion factor of a tentlike map and its topological entropy. To see a simple example of this, let f be the complex map $z^2 + i$ restricted to its Julia Set. Although $h(f) = \log 2$, its tentlike expansion factor will be $\lambda < 2$, where $\log\lambda$ is the topological entropy of f restricted to the Hubbard Tree. However, this is not the whole picture, because it will often be the case when D''_τ is not a tree that the log of the expansion factor is not the same as the topological entropy of σ''_τ, so just taking the (exponential of the) entropy restricted to the smallest continuum containing the orbit will not give the expansion factor in general. (To see this, note that $D_\tau = D''_\tau$ for a dense set of τ, and then look at Theorem 6.33 below.)

The expansion factor of a tentlike map (which, as we show below, is unique) is more closely related to what we might call "linear entropy". For a unimodal dendrite map (or even one with finitely many turning points), this might be defined as the supremum over all subarcs A of $\limsup \frac{1}{n} \log N(f^n, A)$, where $N(g, A)$ is the number of turning points of g which lie in the arc A (see Theorem 6.9 below). In the case of unimodal tree maps, where every arc has interior, this is just the topological entropy, but it will often be strictly smaller than the entropy in general.

Definition 6.1. If $f : D \to D$ is a dendrite map, and d is a metric on D compatible with the topology of D such that for some constant $\lambda \geqslant 1$, $d(f(x), f(y)) = \lambda d(x, y)$ whenever x and y are in the same component of $D \setminus \{t\}$ (where t is the turning point of f), i.e., if d is a metric witnessing that f is tentlike, then d will be called a *tent metric* for f. A function $d : D \times D \to \mathbb{R}$ will be called a *tent pseudometric* with expansion factor $\lambda \geqslant 1$ if and only if the following properties hold:

(1) $d(x, y) \geqslant 0$ for all $x, y \in D$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in D$.
(3) If $y \in [x, z]$, then $d(x, z) = d(x, y) + d(y, z)$.
(4) $d(f(x), f(y)) = \lambda d(x, y)$ whenever x and y are in the same component of $D \setminus \{t\}$.
(5) d is bounded.

A tent metric d is *trivial* if $d(x, y) = 0$ for all $x, y \in D$ and *nontrivial* otherwise. Note that the triangle inequality follows easily from (3) and unique arcwise connectedness, so that if a tent pseudometric has the additional property that equality for (1) holds iff $x = y$, then d will be a metric, and therefore a tent metric provided that it generates the topology of D (which we do not require in the definition). Also, note that if $d(x, z) = 0$, then $d(x, y) = d(y, z) = 0$ for all $y \in [x, z]$, so that the set of all points having pseudodistance 0 from a point x will be a connected (but not necessarily closed) set.

If d is a tent pseudometric on a dendrite D, we let M_d be the supremum of $\{d(x, y)\colon x, y \in D\}$, and if A is an arc in D, we define the *length* of A (with respect to d) to be $d(a, b)$, where $A = [a, b]$.

The following examples show two distinctly different types of tent pseudometrics which are not tent metrics. However, we are primarily interested in such examples for the purpose of avoiding them.

Example 6.2. Let $f : [0, 1] \to [0, 1]$ be any tentish interval map for which the central fixed point c has no preimage other than itself (e.g., a minimally tentish map for $\tau = \overline{012111}$). Then let c have pseudodistance 1 from all other points, and let the remaining pairs of points have pseudodistance either 0 or 2, depending on whether they are on the same or opposite components of c. Then d is a tent pseudometric for f with expansion factor $\lambda = 1$.

Example 6.3. Suppose that σ_τ is tentlike and that $\theta \neq \tau$ is an acceptable sequence such that there is a monotone semiconjugacy $\pi : D_\theta \to D_\tau$ (for example, if τ is a left $\star$-factor of θ). Let d be a tent metric witnessing that σ_τ is tentlike. Then the function d' on $D_\theta \times D_\theta$ defined by $d'(x, y) = d(\pi(x), \pi(y))$ will be a tent pseudometric (with the same expansion factor) which is not a tent metric.

Lemma 6.4. *Let $f : D \to D$ be a tentish dendrite map, and suppose that $d : D \times D \to \mathbb{R}$ is a nontrivial tent pseudometric for f with expansion factor λ. Then d is continuous if and only if $\lambda > 1$.*

Proof. ($\Rightarrow$) By contradiction. Let $\tau = \tau(f)$. Suppose that d is continuous and that $\lambda = 1$. Let $k = k(f)$, $c = c(f)$, and let t be the turning point of f, with $t_n = f^n(t)$ for $n \in \omega$. Define t_n for negative integers n by letting $t_n \in L_1$ be the point with itinerary $1^n\tau$, noting that t_n approaches c as n approaches $-\infty$, and that if $n \leqslant 0$, then $t_n \in [c, t_{n+k}]$. Since $\lambda = 1$ and $[c, t_i]$ is contained in the closure of a leg for $i < k$, we must have $d(c, t_i) = d(c, t_j)$ for all $i < j < k$. Thus, for all $i \leqslant 0$, $d(t_i, t_{i+k}) = 0$, so we must have $d(t, x) = 0$ for all $x \in (c, t)$, and therefore by continuity of d, $d(x, y) = 0$ for all $x, y \in [c, t]$. Again using the fact that $\lambda = 1$, we must have $d(x, y) = 0$ for all $x, y \in \bigcup_{n\in\omega}[c, t_n]$. If x and y are not endpoints of D, then $f^i[x, y]$ is contained in $\bigcup_{n\in\omega}[c, t_n]$ for some i, and we must therefore have $d(x, y) = 0$ for all non-endpoints $x, y \in D$. Continuity then gives us that d must be the identically zero function, a contradiction.

($\Leftarrow$) By contradiction. Suppose that $\lambda > 1$, and let $(a, b) \in D \times D$ be a point at which d is not continuous. Using the fact that d is bounded, let M be a positive real number so that $d(x, y) < M$ for all $x, y \in D$. Then there exists a sequence of pairs (a_n, b_n) from $D \times D$ converging to (a, b) and an $\varepsilon > 0$ such that $|d(a_n, b_n) - d(a, b)| > \varepsilon$ for all $n \in \omega$. Then the taxicab property implies either that some subsequence of the $d(a_n, a)$'s or that some subsequence of the $d(b_n, b)$'s fails to converge to 0. Thus, letting c be either a or b, we have a point $c \in D$ and a sequence of points c_n converging to c such that $d(c_n, c) > \varepsilon > 0$ for all $n \in \omega$ (perhaps a different ε), and by taking a subsequence if necessary, it can be arranged that c and all of the c_n's are in the closure of the same pseudoleg. Thus $d(f(c_n), f(c)) > \lambda\varepsilon$ for all n. Let j be such that $\lambda^j\varepsilon > M$. Then repeating the argument j times (taking a subsequence contained in the closure of a single pseudoleg at each step), we get that $d(f^j(c_n), d(f^j(c)) > \lambda^j\varepsilon > M$ for infinitely many n, contradicting the definition of M. $\square$

Theorem 6.5. *Let $f : D \to D$ be a tentish dendrite map, and let d be a tent pseudometric for f. Then the following are equivalent.*

(1) *d is a tent metric for f.*
(2) *d is a metric.*
(3) *$d(x, y) \neq 0$ for all $x \neq y$ in D.*

Proof. Since (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) is trivial, we only need to prove that (2) implies (1). Thus, suppose that d is a metric and that λ is the expansion factor of d. First, we note that d must be continuous, for otherwise we would have $\lambda = 1$, and the argument of Lemma 6.1 would then give us that $d(t, f^k(t)) = 0$, contradicting that d is a metric. Let D^* be the space with the same underlying set as D, endowed with the topology induced by d. Then continuity of d implies that the identity map from D to D^* is a continuous bijection, and therefore a homeomorphism, since D is compact and D^* is Hausdorff. Thus, the topology induced by d is the same as the Dendrite topology of D. □

Proposition 6.6. *If d is a tent metric for a tentish map $f : D \to D$ with expansion factor λ, then $1 < \lambda \leqslant 2$.*

Proof. $\lambda > 1$ has already been shown, since d is continuous. To see that λ cannot be greater than 2, pick $a, b \in D$ such that $d(a,b)$ is as large as possible. Then $\lambda d(a,b) \leqslant \lambda d(a,t) + \lambda d(t,b) = d(f(a), f(t)) + d(f(t), f(b)) \leqslant 2d(a,b)$. (Note that the tent maps on the interval show that all expansion in the interval $(1,2]$ are possible.) □

Corollary 6.7. *If D''_τ is tentlike with tent metric d and expansion factor λ, and q is any positive integer such that $\{0, 1, 2, \ldots, q\}$ contains the range of τ, then d can be uniquely extended to a tent metric* (*with the same expansion factor*) *on $D_{(q,\tau)}$.*

Proof. Obviously, any such extension would have to have the same expansion factor. Working in $D_{(q,\tau)}$, for each positive integer n, let $X_n = \sigma^{-n}(D''_\tau)$. Then each X_n is a subdendrite of $D_{(q,\tau)}$, $X_n \subseteq X_{n+1}$, and $\bigcup_{n\in\omega} X_n$ contains all non-endpoints of $D_{(q,\tau)}$ (the latter by Proposition 2.34). Working by induction, suppose that d can be uniquely extended to a tent metric on X_n (which we shall still call d). Note that this is trivially true of $n = 0$. Let $\alpha, \beta \in X_{n+1}$. There are four essentially different cases (all other cases being symmetric to one of the ones listed):

Case 1: $\alpha, \beta \in X_n$. Then $d(\alpha, \beta)$ is already defined.

Case 2: $\alpha \in X_n$, $\beta \notin X_n$. Then $X_n \cap [\alpha, \beta] = [\gamma, \beta]$ for some $\gamma \in X_n$, and since $\sigma[\gamma, \beta] \subseteq X_n$, and σ is one-to-one on $[\gamma, \beta]$, we can define $d(\gamma, \beta) = \frac{1}{\lambda} d(\sigma(\gamma)\sigma(\beta))$, and then let $d(\alpha, \beta) = d(\alpha, \gamma) + d(\gamma, \beta)$.

Case 3: $\alpha, \beta \notin X_n$ and α and β are in the same component of $D_{(q,\tau)} \setminus X_n$. Then $\sigma[\alpha, \beta] \subseteq X_n$, and we can let $d(\alpha, \beta) = \frac{1}{\lambda} d(\sigma(\alpha), \sigma(\beta))$.

Case 4: $\alpha, \beta \notin X_n$ and α and β are in different components of $D_{(q,\tau)} \setminus X_n$. Then let $\gamma \in [\alpha, \beta] \cap X_n$, and then let $d(\alpha, \beta) = d(\alpha, \gamma) + d(\gamma, \beta)$, the latter two of which have already been defined in Case 2, and it is easy to see that this is independent of the choice of γ.

The metric d has now been extended to all non-endpoints of $D_{(q,\tau)}$ (and perhaps some endpoints as well). In order to extend to all of $D_{(q,\tau)}$ by simply taking limits at endpoints, all we need to do is show that d is bounded on $\bigcup_{n\in\omega} X_n$. Thus, let M be a positive real number such that $d(\alpha, \beta) < M$ for all $\alpha, \beta \in D_\tau$. Let $\gamma \in \bigcup_{n\in\omega} X_n$. Then there are $\gamma_i \in [\tau, \gamma]$ (perhaps not all distinct) such that $\gamma_0 = \tau$, $\gamma_{n+1} = \gamma$, and γ_i is on the boundary of X_{i-1} for $1 \leqslant i \leqslant n$. Since σ^i is one-to-one on $[\gamma_i, \gamma_{i+1}]$, and $d(\sigma(\gamma_i), \sigma(\gamma_{i+1})) < M$, $d(\gamma_i, \gamma_{i+1}) < M\lambda^{-i}$. Thus, $d(\alpha, \tau) < M \sum_{i=0}^{n} \lambda^{-i}$, and therefore for any $\alpha, \beta \in \bigcup_{n\in\omega} X_n$, $d(\alpha, \beta) \leqslant M \sum_{i=0}^{\infty} \lambda^{-i}$ (a convergent geometric series). Finally, uniqueness of the extension of d is clear from the construction. □

Corollary 6.8. *A tentish map $f : D \to D$ is tentlike if and only if its restriction to a minimally tentish map is tentlike, and thus whether or not a tentish map is tentlike depends only on its kneading sequence.*

Theorem 6.9. *Let $f : D \to D$ be a unimodal tentlike dendrite map with expansion factor λ. Fix an interval $I \subseteq D$ and for each positive integer n, let w_n be the number of turning points of $f^n|I$. Fix $\varepsilon > 0$. Then there are positive real numbers c_1 and c_2* (*which depend on I and ε*) *such that $c_1\lambda^n < w_n < c_2\lambda^n(1+\varepsilon)^n$ for all positive integers n.*

Proof. Let d be a tent metric for f with expansion factor λ, and let $M = M_d$. Let a be the length of I, and let $0 < c_1 < \frac{a}{M}$. Then for each positive integer n, I contains at least $c_1\lambda^n$ pairwise disjoint intervals, each having length $M\lambda^{-n}$, each of which must contain a turning point of f^n. For the other inequality, there are two cases, depending on whether or not the turning point t is periodic.

Case 1: t is periodic under f. Then the number of turning points of f^n is bounded by the sum of the coordinates of the vector $A^n\bar{1}$, where A is the incidence matrix of the Markov Graph of f, and $\bar{1}$ is the vector whose coordinates are all 1's. This is clearly dominated by $c_2\lambda^n(1+\varepsilon)^n$ for some constant c_2, since λ is the dominant eigenvalue of A.

Case 2: t is not periodic under f. Pick a positive integer N such that $\sqrt[N]{2} < 1 + \varepsilon$. Since t is not periodic, and f^N has finitely many turning points, we can pick a positive number η such that no interval of length less than or equal to η has more than one turning point of f^N. Observe that no interval of length $\eta\lambda^{-N}$ has more than three turning points of f^{2N}, for if J is such an interval, then it has no more than one turning point of f^N, and can therefore be divided into two intervals J_1 and J_2 on which f^N is one-to-one, and $f^N(J_1)$ and $f^N(J_2)$, each having length no more than η, and therefore no more than one turning point of f^N, leaving nor more than three possible turning points in J for f^{2N} (at most an original one plus preimages of two turning points from $f(J_1)$ and $f(J_2)$). Repeating this observation, it is easy to prove by induction that any interval of length $\eta\lambda^{-jn}$ has no more than $2^{j+1} - 1$ turning points of f^{jN}. Then I can be written as the union of r_j intervals of length $\eta\lambda^{-jn}$ (where r_j is the least integer greater than or equal to $\frac{a}{\eta}\lambda^{jN}$), each of which has no more than $2^{j+1} - 1$ turning points of f^{jN}, so that there are fewer than $2^{j+1}r$ turning points of f^{jN} in the interval I. Observing that $w_n \leqslant w_{n+1}$ and that $a\lambda^{jN}/r_j\eta$ approaches 1 as j gets large, it is routine to see that there is a constant c_2 so that $w_n < c_2\lambda^n(\sqrt[N]{2})^n < c_2\lambda^n(1+\varepsilon)^n$ for all positive integers n. □

Corollary 6.10. *If f is a unimodal tentlike dendrite map, then the expansion factor λ is unique.*

Theorem 6.11. *Suppose that $\alpha, \beta \in \Gamma$, α is simple with period $n > 1$, and β is acceptable. Then $\sigma_{\alpha\star\beta}$ is tentlike with expansion factor λ if and only if σ_β is tentlike with expansion factor λ^n.*

Proof. Let $\tau = \alpha \star \beta$.

($\Rightarrow$) Let d be a tent metric for σ_τ with expansion factor λ. Since, $\tau = \alpha \star \beta$, σ_τ is renormalizable, and there is a subdendrite $D \subseteq D_\tau$ on which $\sigma^n|D$ is tentish with kneading sequence β. Clearly the same d is a tent metric for $\sigma^n|D$ with expansion factor λ^n.

($\Leftarrow$) Let $D_n = D_0, D_1, \ldots, D_{n-1}$ be n subdendrites of D_τ having no more than the point $\bar{1}$ in common such that $\sigma(D_i) = D_{i+1}$ for $0 \leqslant i \leqslant n-1$, and such that the restriction of σ^n to each to each is a tentish. Without loss of generality (by replacing each D_i with $[D_i \cup \{\bar{1}\}$ if necessary), we may assume that $\bar{1}$ is an element of each D_i. We may also assume that D_0 is the subdendrite containing τ, so that $\sigma^n|D_0$ will have kneading sequence β (using the same labeling as D_τ), and will therefore be tentlike with expansion factor λ^n. Let d be a tent metric for σ^n on $D_0 = D_n$ with that expansion factor. By backwards induction from $n-1$ to 1, define d on D_i by $d(\gamma, \delta) = \lambda^{-1}d(\sigma(\gamma), \sigma(\delta))$, and then extend d to a tent metric all of $D \times D$ in the obvious way (where D is the union of the D_i's, and obviously contains at least D_τ''). □

Corollary 6.12. *Suppose that $\alpha, \beta \in \Gamma$, α is semisimple with period $n > 1$, and β is acceptable. Then $\sigma_{\alpha\star\beta}$ is tentlike with expansion factor λ if and only if σ_β is tentlike with expansion factor λ^n.*

Corollary 6.13. *If f is a tentlike dendrite map that is n-times renormalizable with expansion factor λ, then $\lambda^{2^n} \leqslant 2$.*

Corollary 6.14. *If f is an infinitely renormalizable tentish dendrite map, then f is not tentlike.*

There is more than one way to construct the tent metrics that we are seeking. The fact that every transitive unimodal tree map is tentlike is a trivial consequence of a result of Parry [15]. Another method uses a trick involving the Perron–Frobenius Theorem to get started. Because an infinite dimensional version of the Perron–Frobenius Theorem will be used to show that the constructed tent metrics are unique up to scale, we shall use the latter approach. The finite-dimensional version of the Perron–Frobenius Theorem that will be most directly useful to us can be stated as follows.

Definition 6.15. If G is a directed graph, then G is said to be *strongly connected* if and only if for any two vertices v and w in G, there is a path from v to w which follows the edges of the graph (in the correct direction). If A is an $n \times n$ matrix, define the *incidence graph* of A to be the graph with vertices $1, 2, 3, \ldots, n$ in which there is an edge from i to j if and only if $A_{ij} \neq 0$. The same definition is used for matrices and graphs of infinite size.

Theorem 6.16 *(Perron–Frobenius Theorem). Let A be a matrix containing no negative entries such that the incidence graph of A is strongly connected. Then A has an eigenvalue $\lambda > 0$ such that*

(1) *λ has an eigenvector $\bar{x} > 0$.*
(2) *λ is a geometrically simple eigenvalue, i.e., if $\bar{x}$ and $\bar{y}$ are eigenvectors of A with eigenvalue λ, then $\bar{x}$ and $\bar{y}$ are scalar multiples of each other.*
(3) *If μ is any eigenvalue of A other than λ, then $|\mu| \leqslant \lambda$, and every eigenvector with eigenvalue μ has at least one negative entry.*
(4) *If in addition A has all positive entries, then so does some eigenvector of λ.*

Corollary 6.17. *If τ is a prime acceptable periodic sequence, then σ''_τ is tentlike, with a unique expansion factor λ, and with a tent metric that is unique up to scale (i.e., if d and d' are two tent metrics for σ_τ, then there is a positive constant a such that $d'(\alpha, \beta) = ad(\alpha, \beta)$ for all $\alpha, \beta \in D''_\tau$).*

Proof. We give a basic outline of the well-known construction. If we let Z be the set of all points which are either in the orbit of τ or are a branching point of D''_τ, then Z defines a finite Markov partition of D''_τ. Enumerating the closures of the components of $D''_\tau \setminus Z$, we get arcs $I_1, I_2, \ldots, I_n$ (which are the vertices of out Markov Graph G) and a relation $I_i \to I_j$ iff $I_j \subseteq \sigma(I_i)$ (the edges of G). The 0, 1-matrix A defined by $A_{ij} = 1$ iff $I_i \to I_j$ is then easily seen to be strongly connected (since τ prime implies σ''_τ strongly transitive) giving the desired eigenvalue λ with unique (up to scale) eigenvector $\bar{v} \geqslant \bar{0}$. Since $\bar{v}$ is also an eigenvector of A^m for all m, strongly arc transitive implies A^m has all positive entries for some m and therefore so does $\bar{v}$. Assigning I_j to have length v_j and defining d at all other points in the obvious way (first on the dense set $Pre_f(Z)$ by induction on f^{-n}) gives the desired tent metric. Uniqueness follows easily from the fact that for any other tent metric, letting u_j be the length of I_j gives a positive eigenvector $\bar{u}$ of A which must then be a positive multiple of $\bar{v}$. $\square$

The generalization to the case in which τ is not periodic is more complicated. The most difficult part of the proof will be the uniqueness up to scale, for which we use an infinite-dimensional version of the Perron–Frobenius Theorem. We remind the reader of some basic definitions regarding the Banach spaces ℓ^p (with $p = 1$ and $p = \infty$ being the only cases of interest to us).

Definition 6.18. ℓ^1 is the space of all sequences $\bar{x} = \langle x_0, x_1, \ldots \rangle$ of real numbers whose sums converge absolutely, with norm $\|\bar{x}\|_1 = \sum_{i=0}^{\infty} |x_i|$, and ℓ^∞ is the space of all bounded sequences of real numbers, with norm $\|\bar{x}\|_\infty = \sup_{i \in \omega} |x_i|$. If $\bar{x}$ and $\bar{y}$ are sequences (finite or infinite) of real numbers having the same length, we define $\bar{x} \geqslant \bar{y}$ to mean that $x_i \geqslant y_i$ for all i such that x_i and y_i are defined, and the same for $\leqslant$. We define $\bar{x} > \bar{y}$ to mean that $\bar{x} \geqslant \bar{y}$ and $\bar{x} \neq \bar{y}$ (i.e., $x_i \geqslant y_i$ for all relevant i and $x_i > y_i$ for at least one i), and $\bar{x} \gg \bar{y}$ means that $x_i > y_i$ for all relevant i, and similarly for $<$ and $\ll$. In addition, we define $|\bar{x}|$ to be the vector in which each entry of $\bar{x}$ is replaced by its absolute value, and similarly for $|A|$ where A is a matrix. We define a natural embedding $e_n : \mathbb{R}^n \to \ell^\infty$ by viewing an element $\bar{x}$ of $\mathbb{R}^n$ as a sequence of length n and letting $e_n(\bar{x})|n = \bar{x}$ and $(e_n(\bar{x}))_i = 0$ for $i \geqslant n$, and $e'_n : \mathbb{R}^n \to \ell^1$ is defined similarly. The $\|.\|_\infty$ and $\|.\|_1$ norms on $\mathbb{R}^n$ are then defined in the obvious way with respect to this embedding, noting that the sets $S_n = \{\bar{x} \in \mathbb{R}^n \colon \|\bar{x}\|_\infty = 1, \bar{x} \geqslant \bar{0}\}$ and $S'_n = \{\bar{x} \in \mathbb{R}^n \colon \|\bar{x}\|_1 = 1, \bar{x} \geqslant \bar{0}\}$ are both homeomorphic to $[0, 1]^{n-1}$, and therefore have the fixed point property by the Brouwer Fixed Point Theorem. We define the map $\pi : \ell^\infty \to \mathbb{R}^n$ by $\pi(\bar{x})|(n-1) = \bar{x}|(n-1)$ and $(\pi(\bar{x}))_{n-1} = \sup_{i \geqslant n-1} x_i$. Similarly, $\pi' : \ell^\infty \to \mathbb{R}^n$ is defined by $\pi'(\bar{x})|(n-1) = \bar{x}|(n-1)$ and $(\pi'(\bar{x}))_{n-1} = \sum_{i=n-1}^{\infty} x_i$. Note that $\pi \circ e$ and $\pi' \circ e'$ are the identity map on $\mathbb{R}^n$. Note also that if A is an ω-by-ω matrix of real numbers such that $|A|$ has uniformly bounded convergent row sums, then $A\bar{x}$ defines a map from ℓ^∞ into itself, and $A^{\mathrm{T}}\bar{x}$ defines a map from ℓ^1 into itself (viewing $\bar{x}$ as a column vector in the obvious way).

The following is a routine generalization of the Perron–Frobenius Theorem, the proof of which was adapted from the topological proof of the finite-dimensional version in [6]. It is not difficult to prove the theorem in more generality, but the one here will suffice for our purpose.

Lemma 6.19. *Let M be a positive integer, and suppose that A is an ω-by-ω matrix of zeros and ones in which each row has no more than M ones, and such that the incidence matrix of A is strongly connected. Then there exists a unique positive eigenvalue λ of A (viewed as defining a linear map from ℓ^∞ into itself) such that λ has an eigenvector with* *no negative entries. Furthermore, λ is a geometrically simple eigenvalue (i.e., any two eigenvectors with eigenvalue λ are scalar multiples of each other), the eigenspace of which is generated by a strictly positive eigenvector $\bar{x} \gg \bar{0}$.*

Proof. Let $S = \{\bar{x} \in \ell^\infty\colon \|\bar{x}\|_\infty = 1, \bar{x} > \bar{0}\}$. Note that if $\bar{x} > \bar{0}$, then strong connectedness implies that every row and column of A has at least one nonzero entry, so that $A\bar{x} > \bar{0}$. Thus, we can define the map $f : S \to S$ by $f(\bar{x}) = \|A\bar{x}\|_\infty^{-1} A\bar{x}$. For each n, the composition $\pi_n \circ f \circ e_n$ is a continuous map from S_n into itself, and therefore has a fixed point $\bar{x}_n$ by the Brouwer Fixed Point Theorem, and it is easy to check that the points $e_n(\bar{x}_n)$ have a subsequence which converges pointwise to a fixed point $\bar{x}$ of f. Letting $\lambda = \|A\bar{x}\|_\infty$, we see that $A\bar{x} = \lambda\bar{x}$. Since $\bar{x}$ has no negative entries, we see that if $x_j > 0$ and $i \to j$ in the incidence graph of A, then we must also have $x_i > 0$. Since the incidence graph is strongly connected and $\bar{x}$ has at least one positive entry, we see that $\bar{x} \gg \bar{0}$. Applying the same argument with A^{T} and ℓ^1, we get a positive eigenvalue λ' of A^{T} having a strictly positive eigenvector $\bar{y} \in \ell^1$. Then $\lambda'\bar{y}^{\mathrm{T}}\bar{x} = \bar{y}^{\mathrm{T}}A\bar{x} = \lambda\bar{y}^{\mathrm{T}}\bar{x}$, and since $\bar{y}^{\mathrm{T}}\bar{x} \neq 0$, we must have $\lambda' = \lambda$. The same argument then shows that no other eigenvalue of A has an eigenvector with nonnegative entries.

To show that every eigenvector with eigenvalue λ is a scalar multiple of $\bar{x}$, we suppose that there is a $\bar{x}' \in \ell^\infty$ such that $A\bar{x}' = \lambda\bar{x}'$ and $\bar{x}'$ is not a scalar multiple of $\bar{x}$, and aim for a contradiction. Then there is some $\bar{v}$ which is a linear combination of $\bar{x}$ and $\bar{x}'$ and $\bar{v}$ has both positive and negative entries. By strong connectedness of the incidence graph of A, there are i and j such that $i \to j$ in the incidence graph, $v_j > 0$, and $v_i \leqslant 0$. Since $A\bar{v} = \lambda\bar{v}$, we have that $\lambda v_i = \sum_{i\to k} v_k$, so the sum on the right-hand side must contain both positive and negative entries, so that we have $|\lambda v_i| < \sum_{itok} |v_k|$ and therefore $A|\bar{v}| > \lambda|\bar{v}|$, and thus $\bar{y}^{\mathrm{T}}A|\bar{v}| > \bar{y}^{\mathrm{T}}(\lambda|\bar{v}|)$ (since $\bar{y}^{\mathrm{T}}$ has all positive entries). However, this contradicts the fact that $\bar{y}^{\mathrm{T}}A|\bar{v}| = (\lambda\bar{y}^{\mathrm{T}})|\bar{v}|$, and we are done. □

Theorem 6.20. *If τ is prime, and d and d' are any two tent metrics for D''_τ, then d and d' are constant multiples of each other.*

Proof. By contradiction. Suppose that d and d' are two tent metrics for σ''_τ which are not constant multiples of each other. By multiplying one of them by a constant if necessary, we may assume that $d(\bar{1}, \tau) = d'(\bar{1}, \tau)$. Note that if $d|I^2 = d'|I^2$ for any interval $I \subseteq D''_\tau$, then $d|(\sigma(I))^2 = d'|(\sigma(I))^2$, so we cannot have $d|[\bar{1}, \tau]^2 = d'|[\bar{1}, \tau]^2$, for otherwise we would then have $d = d'$. Thus, there is a $\alpha \in [\bar{1}, \tau]$ such that $d(\bar{1}, \alpha) \neq d'(\bar{1}, \alpha)$, and since the periodic points are arc-dense in D''_τ when τ is prime, we may, by moving α a small distance if necessary, assume that α is periodic. Let $S = \{\bar{1}, \tau\} \cup Orb_\sigma(\alpha)$ and $T = S \cup Orb_\sigma(\tau)$. Let W be the set of all arcs $I \subseteq D''_\tau$ such that the endpoints of I are members of T and the interior of I misses S. We form a directed graph G with W as the vertex set by defining $I \to J$ if and only if J is the closure of one of the components of $f(I) \setminus S$, noting that for a given $I \in W$, there cannot be more than M elements $J \in W$ such that $I \to J$, where $M - 1$ is the number of elements of S. Let $\beta \in S$ be such that $[\bar{1}, \beta] \in W$ and $\beta \in [\bar{1}, \tau]$. There is clearly only one such β. Let $K = [\bar{1}, \beta]$.

Claim. *For any $I \in W$, there is a path in G from I to K.*

Proof. Since τ is prime, σ''_τ is strongly arc-transitive (Theorem 4.13), and thus $\bar{1} \in \sigma^n(I)$ for some n, from which it follows that there is a $[\bar{1}, \gamma] \in W$ such that $[\bar{1}, \gamma]$ can be reached from I in G by a path of n or fewer steps. Since it is easy to see that there is a finite path in G from $[\bar{1}, \gamma]$ to K, this finishes the claim. □

Now, let V be the set of all members of W which can be reached in G from K by a finite path, and let H be the subgraph of G with vertex set V. Then the claim implies that H is strongly connected. Enumerate the members of V as I_n, $n \in \omega$, and form the ω-by-ω matrix A of zeros and ones by letting $A_{ij} = 1$ if and only if $I_i \to I_j$. Define $\bar{x} \in \ell^\infty$ by letting x_n be the length of the interval I_n with respect to the metric d, and define $\bar{x}'$ in the same way using d'. Then $\bar{x}$ is a strictly positive eigenvector of A with eigenvalue λ, where λ is the expansion factor of d, and the same holds for $\bar{x}'$. Since the hypotheses of Lemma 6.16 hold, we must have that $\bar{x}$ and $\bar{x}'$ are constant multiples of each other. However, since $[\bar{1}, \tau)]$ and $[\bar{1}, \alpha)]$ are both finite unions of nonoverlapping (except at endpoints) members of V, this contradicts the assumption that $d(\bar{1}, \tau) = d'(\bar{1}, \tau)$ and $d(\bar{1}, \alpha) \neq d'(\bar{1}, \alpha)$. □

Lemma 6.21. *Let $f : D \to D$ be a continuous dendrite function, and let $a_i, b_i \in D$ $(0 \leqslant i \leqslant n)$ be such that $a_i \neq b_i$, $f(a_i) = a_{i+1}$ and $f(b_i) = b_{i+1}$ for $0 \leqslant i \leqslant n-1$, $a_0 \in (a_n, b_0)$, and $b_0 \in (a_0, b_n)$. Then (a_0, b_0) contains a fixed point x of f^n such that $f^i(x) \in (a_i, b_i)$, $0 \leqslant i \leqslant n$.*

Proof. Let $r_i : D \to [a_i, d_i]$ be the natural retraction for D to $[a_i, b_i]$ (i.e., $r_i(x)$ is the unique point r such that $[x, r] \cap [a_i, b_i]$ is a singleton). Let $g : [a_0, b_0] \to [a_n, b_n]$ be defined by

$$g = r_n \circ f \circ r_{n-1} \circ f \circ \cdots \circ r_1 \circ f.$$

Then since $[a_0, b_0] \subseteq [a_n, b_n]$, g has at least one fixed point, and the fixed point of g closest to a_0 (or to b_0) will be the desired point. (If a fixed point fails to satisfy the desired properties, then it is easy to show there was another fixed point of g even closer to a_0.) See [2] for a more detailed proof. $\square$

Lemma 6.22. *Let $f : D \to D$ be any continuous dendrite map, let $t \in D$, and let A and B be two distinct components of $D \setminus \{t\}$. Suppose that $f^n(t) \in B$ and that $f^i(t) \in A$ for $1 \leqslant i \leqslant n-1$. Let E be the smallest subtree of D containing the points of $C = \{t, f(t), f^2(t), \ldots, f^{n-1}(t)\}$. Then there is a closed set $K \subseteq E$ such that $f(K) \subseteq K$ and no two points of C are in the same component of $E \setminus K$.*

Proof. Let $y_i = f^i(t)$ for $0 \leqslant i \leqslant n-1$, and for convenience, let $y_i = y_{n-i}$ for all $i \geqslant n$. Fix $0 \leqslant j < k \leqslant n-1$. Let $a_n = y_{j+n}$, $b_n = y_{j+n}$. By backwards induction on $i < n$, define $a_i, b_i \in [y_{j+i}, y_{k+i}]$ such that $f(a_i) = f(a_{i+1})$, $f(b_i) = f(b_{i+1})$, $a_i \in [y_{j+i}, b_i]$, and $b_i \in [a_i, y_{k+i}]$. Then a_i, b_i satisfy the hypothesis of the previous lemma, so there is a fixed point x of f^n whose entire orbit is contained in the tree E. If we let K be the set of all fixed points of f^n whose entire orbit is in E, then K is as desired. $\square$

Lemma 6.23. *Let τ be acceptable, let $s \geqslant 1$, $j = \tau_s \neq 0$, and suppose $m > s$ is least such that $\tau_{ms} \notin \{0, j\}$, and $n > m$ is such that n is not a multiple of m and $\tau_{ns} \notin \{0, j\}$. Suppose in addition that d is a tent metric for D''_τ with expansion factor λ. Then $\lambda^p > 2$, where p is the larger of ms and $(n-m)s$.*

Proof. Note that if I and J are intervals such that $J \subseteq \sigma^i(I)$, then the length of J can be no more than λ^i times the length of I. Let $C = \{\tau, \sigma^s(\tau), \sigma^{2s}(\tau), \ldots, \sigma^{(m-1)s}(\tau)\}$, noting that $C \subseteq L_j \cup \{\tau\}$. Since the hypothesis of the previous lemma holds for the points in C and the function σ^s, there is a σ^{ms}-invariant set $K \subseteq [C]$ which separates the elements of C. Let a be the taxicab distance between the sets C and K, and pick and interval I if length a having one endpoint in C and the other in K. Let $J = [\tau, \sigma^{ms}(\tau)]$.

Case 1: J has d-length greater than a. Then for some $r \leqslant m$, $\sigma^{ms}(\tau) \in \sigma^{rs}(I)$, so that $\sigma^{rs}(I)$ contains both J and an interval having one endpoint in C and the other in K (therefore of d-length at least a), so that $\lambda^{ms} \geqslant \lambda^{rs} > 2$ (since the two intervals do not overlap).

Case 2: J has d-length a or less. Then since n is not a multiple of m, $\sigma^{(n-m)s}(\tau)$ is an element β of C other than τ, so $\sigma^{(n-m)s}(J)$ contains both β and $\sigma^{ns}(\tau) \notin L_j \cup \{\tau\}$, and therefore contains at least two nonoverlapping intervals from C to K (each of length at least a), plus at least a little more outside of $L_j \cup \{\tau\}$, and so has d-length strictly greater than $2a$. Thus $\lambda^{(n-m)s} > 2$. $\square$

Corollary 6.24. *Let τ be acceptable, and suppose that there exists a positive integer n such that n is not a multiple of $k(\tau)$ and $\tau_n \neq \tau_1$. Suppose in addition that d is a tent metric for D''_τ with expansion factor λ. Then $\lambda^p > 2$, where p is the larger of k and $n - k(\tau)$.*

Proof. Let $s = 1$ in the previous lemma. $\square$

Corollary 6.25. *Let τ be pseudoprime and periodic with period n, and suppose that d is a tent metric for D''_τ with expansion factor λ. Then $\lambda^n > 2$.*

Proof. Since τ is pseudoprime and periodic, then τ must be of the form $\tau = \alpha \star \beta$, where α and β are periodic, α is semisimple, and β is prime. It is then a simple proof by induction on the number of simple factors of α, using the previous lemma if $\alpha = \bar{0}$, and Theorem 6.11 otherwise. $\square$

Lemma 6.26. *Let τ be acceptable, and let α be a finite sequence from range(τ) of length n such that $\tau \in B_{\sigma^i(\alpha)}$ for any i such that $\alpha_i = 0$. Then the closure (in D_τ) of $B_\alpha \cap D_\tau$ is connected.*

Proof. By induction on the length of α. The case where α has length 1 is clear, since $L_{\alpha_1} \cup \{\tau\}$ is connected. If $\overline{B_\alpha}$ is connected and $1 \leqslant j \leqslant q$, then $\sigma|B_{j\alpha}$ is a homeomorphism from $\overline{B_{j\alpha}}$ onto $\overline{B_\alpha}$. As for the case $j = 0$, $\tau \in B_{0\alpha}$ implies $\sigma(\tau) \in B_\alpha$, which, by Theorem 2.35, implies that $\overline{B_{0\alpha}} = \overline{\sigma^{-1} B_\alpha}$ is connected. □

Note that the closures would be unnecessary if we were dealing with D'_τ or D''_τ, since we would then not have to worry about nontrivial pseudolegs.

Corollary 6.27. *Let τ be acceptable, let d be a tent metric with expansion factor λ, and let α be a finite sequence from range(τ) of length n such that $\tau \in B_{\sigma^i(\alpha)}$ for any i such that $\alpha_i = 0$. Then $B_\alpha \cap D_\tau$ has diameter no more than $2^j M_d/\lambda^n$, where j is the number of coordinates on which $\alpha_i = 0$.*

Proof. Since $\overline{B_\alpha}$ is connected by the lemma, given $\beta, \gamma \in B_\alpha$, there is an arc $A \subseteq \overline{B_\alpha}$ containing both β and γ. Then σ is one-to-one on $\sigma^{j-1}(A)$ for $n - j$ of the applications of σ and at most two-to-one for the other j applications. Thus $\sigma^n(A)$ is the union of no more than 2^j arcs in D''_τ, each of which is a preimage under σ^n of an arc of length no more than M_d/λ^n. □

Note that the condition on α is necessary, for if $\beta, \gamma \in D''_\tau$ are distinct such that $\delta = \sigma(\beta) = \sigma(\gamma)$, then $\alpha = 0\delta$ would give $\beta, \gamma \in B_{\alpha|N}$ for all N.

Theorem 6.28. *Suppose τ is acceptable but not pseudoprime. Then σ_τ is not tentlike.*

Proof. Since τ is acceptable but not pseudoprime, there is an $\alpha \in \mathcal{L}_q$ and an acceptable β such that $\tau = \alpha \star \beta$.

Case 1: α is prime. Aiming for a contradiction, suppose that σ_τ is tentlike with expansion factor λ and tent metric d. Then by Theorem 3.20, there is a semiconjugacy $\pi : D_\tau \to D_\alpha$ such that π has connected point inverses. By Corollary 6.25, $\lambda^p > 2$, where $p = p(\alpha)$. Let $\beta = \sigma(\alpha)|p$, $\gamma = \overline{1\beta}$, $\delta = \overline{2\beta}$, i.e., γ and δ are the result of replacing all 0's in α by 1 and 2, respectively. Let $c = d(\gamma, \delta)$, and pick j large enough so that $2^j M_d/\lambda^N < c$, where $N = jp$. Then the hypothesis of Corollary 6.27 holds, and $B_{\alpha|N}$ has diameter less that $2^j M_d/\lambda^N < c$, contradicting that γ and δ are both in $B_{\alpha|N}$.

Case 2: α is not prime. Then since α is periodic, $\alpha = \gamma \star \delta$ for some prime δ, and therefore $\tau = \gamma \star \delta \star \beta$. But $\delta \star \beta$ is not tentlike by Case 1, and therefore neither is τ, by Theorem 6.11. □

Lemma 6.29. *If d is a tent metric for $\sigma_{(q,\tau)}$ with expansion factor λ, then $M_d \leqslant \frac{\lambda+1}{\lambda-1} d(\bar{j}, \tau)$, where $j = \tau_1$.*

Proof. Let $A \subseteq D_{(q,\tau)}$ be an arc with length $M = M_d$. Then A cannot be contained in a single leg, because $\sigma(A)$ would then be longer. Thus, $\tau \in A$, and we may assume, without loss of generality, that A has length $M/2$, for otherwise $\sigma^{-1}(\sigma(A))$ would contain a longer arc, because the pseudolegs have identical metrics. Thus, each of the two pieces of A (split by τ) has length $M/2$, and no arc contained in a leg can have any longer length than $M/2$. Thus, if A' is one half of A, then $\sigma(A')$ can have length no longer than $M/2 + a + \lambda a$, where $a = d(\bar{j}, \tau)$ (and $a + \lambda a$ is the length of $[\tau, \sigma(\tau)]$). Thus, $\lambda M/2 \leqslant M/2 + a + \lambda a$, and solving for M gives the result. □

Lemma 6.30. *Let $\tau \in \mathcal{L}_q$, and suppose that $\langle \tau^{(n)} \rangle$ is a sequence from $\mathcal{L}_q$ which converges to τ, such that each $\sigma_{\tau^{(n)}}$ is tentlike with tent metric d_n and expansion factor λ_n, and such that $d_n(\overline{\tau_1}, \tau^{(n)}) = 1$ for all n. For each $\alpha, \beta \in Q^\omega$ (where $Q = \{1, 2, \ldots, q\}$), let $f_n(\alpha, \beta) = d_n(\chi_{\tau^{(n)}}(\alpha), \chi_{\tau^{(n)}}(\beta))$. Then there exists a subsequence $\langle \tau^{s(n)} \rangle$ of the $\tau^{(n)}$'s such that $\lambda_{s(n)}$ converges to some $\lambda > 1$, and the $f_{s(n)}$'s converge uniformly to a function f.*

Furthermore, there is a tent metric d on D_τ with expansion factor λ such that $f(\alpha, \beta) = d(\chi_\tau(\alpha), \chi_\tau(\beta))$ for all $\alpha, \beta \in D_\tau$.

Proof. To avoid trivialities, assume that the sequence $\tau^{(n)}$ is not constant on any subsequence, so that the periods of any $\tau^{(n)}$'s which happen to be periodic (if any) get arbitrarily large as n increases. Let η be semisimple and let θ

be prime such that $\tau = \eta \star \theta$. Let $s = s(\eta)$. If we let $m > k(\theta)$ be least such that nm is not a multiple of $k(\theta)$ and $\theta_m \notin \{0, \theta_1\}$ (which must exist since θ is prime), then the hypotheses of Lemma 6.23 hold for all but finitely many $\tau^{(n)}$'s, so that $\lambda_n^p > 2$ for all but finitely many n for some fixed p, which may be assumed to be the period of τ if τ happens to be periodic. Thus, we may assume (by taking a subsequence if necessary), that the λ_n's converge to some $\lambda \geqslant a = \sqrt[p]{2}$. Let $E = \{e_n\colon n \in \omega\}$ be a dense subset of $Q^\omega \times Q^\omega$. By Lemma 6.29, we know that the M_{d_n}'s are bounded by some fixed M, so the f_n's are bounded, and we can get a further subsequence such that the f_n (in the new subsequence) converge at e_0, another where they converge at e_1, etc., and then use the usual diagonal argument to get a subsequence $\langle f_{s(n)}\rangle$ such that the $f_{(s(n))}$'s converge (pointwise) at every point of E. For convenience, this subsequence will now be renamed as $\langle f_n\rangle$. For any $\alpha \in Q^\omega$, let $\alpha^{(n)}$ abbreviate $\chi_{\tau_n}(\alpha)$, and similarly for $\beta^{(n)}$, $\gamma^{(n)}$, etc.

We now want to show that this subsequence $\langle f_n\rangle$ converges uniformly on all of $Q^\omega \times Q^\omega$. Thus, let $\varepsilon > 0$. Pick $N = N(\varepsilon)$ large enough so that $a^N > 10M/\varepsilon$. Then pick a finite $E(\varepsilon) \subseteq E$ so that every basic open subset of the form $B_\gamma \times B_\delta$ contains a member of $E(\varepsilon)$ for all γ and δ of length $N(\varepsilon)$. Then pick $K = K(\varepsilon)$ large enough so that for every $n \geqslant K(\varepsilon)$, $\tau_i^{(n)} \neq 0$ for $1 \leqslant i N - 1$ and so that for every $m, n \geqslant K(\varepsilon)$, $|f_n(e) - f_n(e)| < \varepsilon/5$.

Then, given any $(\alpha, \beta) \in Q^\omega \times Q^\omega$, and any $m, n \geqslant K(\varepsilon)$, $\chi_{\tau^{(n)}}(\alpha)$ will be 0 on no more than one coordinate, and $B_{\chi_{\tau(n)}(\alpha)}$ will satisfy the hypothesis of Corollary 6.27, and will thus have diameter less than $2M/a^N < \varepsilon/5$, and for the same reason, $B_{\chi_{\tau(n)}(\beta)}$ will have diameter less than $\varepsilon/5$.

Thus, for $m, n \geqslant K$ and $(\alpha, \beta) \in Q^\omega \times Q^\omega$, pick $e = (\gamma, \delta)$ such that $e \in E(\varepsilon)$ and $e \in B_{\alpha|N} \times B_{\beta|N}$. Since $\alpha, \gamma \in B_{\alpha|N}$, and $\alpha_i^{(n)} = 0$ for at most one value of i (since the same is true of τ) we have by Corollary 6.27,

$$d_n(\alpha^{(n)}, \gamma^{(n)}) < 2^{N-1}M/a^N < \varepsilon/5, \qquad d_m(\alpha^{(m)}, \gamma^{(m)}) < 2^{N-1}M/a^N < \varepsilon/5,$$

and similarly for β and δ,

$$d_n(\delta^{(n)}, \beta^{(n)}) < \varepsilon/5, \qquad d_m(\delta^{(m)}, \beta^{(m)}) < \varepsilon/5,$$

and we also have

$$\left|d_n(\gamma^{(n)}, \delta^{(n)}) - d_m(\gamma^{(m)}, \delta^{(m)})\right| = \left|f_n(e) - f_m(e)\right| < \varepsilon/5.$$

Combining these gives

$$\left|f_n(\alpha, \beta) - f_m(\alpha, \beta)\right| = \left|d_n(\alpha^{(n)}, \beta^{(n)}) - d_m(\alpha^{(n)}, \beta^{(n)})\right| < \varepsilon,$$

so that the f_n's are uniformly Cauchy and thus converge uniformly to a continuous function f.

If $\chi_\tau(\alpha) = \chi_\tau(\beta) = \gamma$, then $\alpha, \beta \in B_{\gamma|N}$ for all N. If τ is not periodic, then $\gamma|N$ is zero on at most one coordinate, and if τ is periodic with period p, then $\gamma|M$ is zero on no more than $1 + N/p$ coordinates, so since $\lambda^p > 2$ in the latter case, in either case we have that the $d_n(\alpha^{(n)}, \beta^{(n)})$ converges to 0, so that $f(\alpha, \beta) = 0$. Thus, there is a unique well-defined continuous map $d\colon D_\tau \times D_\tau \to [0, \infty)$ such that $f(\alpha, \beta) = d(\chi_\tau(\alpha), \chi_\tau(\beta))$ for all $\alpha, \beta \in Q^\omega$. It is clear that d is a pseudometric (since all of the d_n's are) and that $d(\sigma(\alpha), \sigma(\beta)) = \lambda d(\alpha, \beta)$ for all $\alpha, \beta \in D_\tau$. The "taxicab" property will hold for d, since if $\chi_\tau(\gamma) \in [\chi_\tau(\alpha), \chi_\tau(\beta)]$ in D_τ, then γ will be approximated in the $D_\tau^{(n)}$'s by sequences $\gamma(n)^{(n)}$ (perhaps different from γ) such that $\chi_{\tau^{(n)}}(\gamma(n)^{(n)})$ is between $\alpha^{(n)}$ and $\beta^{(n)}$ in $D_{\tau^{(n)}}$, an easy consequence of the continuity of the functions $w(\alpha, \beta, \gamma)$ and the fact that only a finite part of the kneading sequences of α, β, γ is needed to get the corresponding finite part of the kneading sequence of $w(\alpha, \beta, \gamma)$ (see the remarks on voting sequences above in Section 3).

To see that d is a metric, note that if τ is prime, then arc-transitivity of σ_τ'' implies that no interval can have length zero with respect to d. If τ is not prime, then the same argument can be used on the prime right factor of τ and a subdendrite, using the appropriate σ^m. □

Corollary 6.31. *If $\tau \in \mathcal{L}_q$, and $j = \tau_1$, then there is a unique tent metric d for σ_τ such that $M_d = 1$.*

Proof. Existence follows from the previous result, and uniqueness has already been proven. □

Definition 6.32. For each $\tau \in \mathcal{L}_q$, let d_τ be the unique tent metric for σ_τ guaranteed by the previous corollary such that $M_d = 1$.

Theorem 6.33. *If $\tau, \tau^n \in \mathcal{L}$ are such that the τ^n's approach τ in the itinerary topology, then the d_{τ_n}'s approach d_τ uniformly in the same sense as in Lemma* 6.27.

Proof. First, note that if $\alpha_1 \neq \beta_1$ and $\alpha, \beta \in \mathcal{L}_q$, then α and β are in different components of $\mathcal{L}_q$. If the theorem failed, then the argument of Lemma 6.26 could be used to get a subsequence approaching a different limit, giving two different tent metrics with $d(\overline{\tau_1}, \tau) = 1$, contradicting uniqueness up to scale. □

Thus, we have the interesting fact that the tentlike dendrite maps are analogous to the interval tent maps in the respect that they have a natural parameterization such that the maps and their tent metrics and expansion factors vary continuously in a parameter space whose one point compactification is itself a dendrite. Note that a simple trick is available to extend this to all of $\mathcal{M}_q$. We could scale the tent metrics instead so that M_d approaches 0 as λ approaches 1 (say, by multiplying each tent metric by the appropriate $\lambda_\tau - 1$), and define D_o as a one point space (with expansion factor 1), in which case the entire structure would be in a compact space.

Index of terms:

Latin notation:

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Further reading

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